Quantum Theory I: Final Exam

19 May 2022

Name: Solution **Total:** /50

Instructions

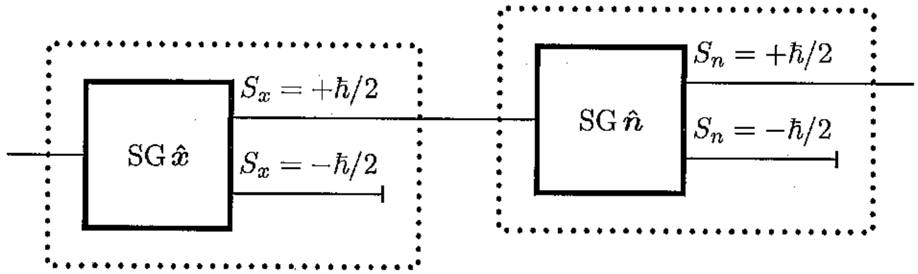
- There are 7 questions on 14 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Planck's constant	$h = 6.63 \times 10^{-34}\,\mathrm{Js}$ $\quad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$
Mass of electron	$m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = 511 \times 10^3\,\mathrm{eV/c^2}$
Mass of proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 938.3 \times 10^6\,\mathrm{eV/c^2}$
Mass of neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg} = 939.6 \times 10^6\,\mathrm{eV/c^2}$
Spherical coordinates	$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$
Spin 1/2 state	$\lvert +\hat{n}\rangle = \cos(\theta/2)\lvert +\hat{z}\rangle + e^{i\phi}\sin(\theta/2)\lvert -\hat{z}\rangle$
Spin 1/2 state	$\lvert -\hat{n}\rangle = \sin(\theta/2)\lvert +\hat{z}\rangle - e^{i\phi}\cos(\theta/2)\lvert -\hat{z}\rangle$
Euler relation	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
Spin observables	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Rep. in $\lvert\pm\hat{\mathbf{z}}\rangle$ basis	$\hat{R}(\varphi \boldsymbol{n}) = \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\cos\theta & -i\sin\left(\frac{\varphi}{2}\right)e^{-i\phi}\sin\theta\\ -i\sin\left(\frac{\varphi}{2}\right)e^{i\phi}\sin\theta & \cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\cos\theta\end{pmatrix}$

Question 1

Spin-1/2 particles are subjected to a sequence of Stern-Gerlach measurements. The first measures S_x and only particles that emerge with $S_x = +\hbar/2$ pass along to the second. The second can be oriented along various directions $\hat{\mathbf{n}}$. The $S_n = -\hbar/2$ output is blocked.



a) Suppose that $\hat{\mathbf{n}} = \hat{\mathbf{y}}$. Consider particles that emerge after the SG $\hat{x}$ device. Determine the probability with which these emerge from the right dotted box.

They emerge in the state $|+\hat{y}\rangle = \cos\frac{\pi}{4}|+\hat{z}\rangle + e^{i\pi/2}\sin\frac{\pi}{4}|-\hat{z}\rangle$ $\quad \theta = \pi/2$, $\phi = \pi/2$

$$= \frac{1}{\sqrt{2}}\left[|+\hat{z}\rangle + i\,|-\hat{z}\rangle\right]$$

Then $\text{prob}(S_x = \pm\hbar/2) = |\langle \pm\hat{x}|\hat{y}\rangle|^2$

Now $|\pm\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle \Rightarrow \langle \pm x|\hat{y}\rangle = \frac{1}{2}(1+i)$

$|\langle \pm\hat{x}|\hat{y}\rangle|^2 = \frac{1}{2}(1+i)\frac{1}{2}(1-i) = \frac{1}{2} \Rightarrow \text{Prob}(S_x = \pm\hbar/2) = 1/2$

b) Suppose that the entire illustrated sequence was followed with one more SG $\hat{x}$ device. Considering all possible choices for $\hat{\mathbf{n}}$ describe whether or not the third measurement would yield $S_x = +\hbar/2$ *with certainty.*

If $\hat{n} = \hat{x}$ then the second will give $S_x = +\hbar/2$ and state $|+\hat{x}\rangle$ with certainty. The third will also do this.

But for all other $\hat{n}$ (except $\pm\hat{x}$) the state $|+\hat{n}\rangle$ is such that $\text{prob}(S_x = +\hbar/2) = |\langle +\hat{x}|+\hat{n}\rangle|^2 \neq 1$.

So it will not occur with certainty unless $\hat{n} = \hat{x}$. /6

b) Suppose that the entire ensemble were subjected to a measurement of $S_{\cancel{x}y}$. Determine the expectation value $\langle S_{\cancel{x}y} \rangle$ provided that every ensemble member were in the state $|\Psi\rangle$.

$$\langle S_y \rangle = \langle \Psi | \hat{S}_y | \Psi \rangle$$

$$\hat{S}_y \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\langle \Psi | = |\Psi\rangle^{+} \mapsto \left[\frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix}\right]^{+} = \frac{1}{5}\begin{pmatrix} 3 & -4i \end{pmatrix}$$

So

$$\langle S_x \rangle = \frac{\hbar}{2}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4i \end{pmatrix}\frac{1}{5^2}$$

$$= \frac{\hbar}{50}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 4 \\ 3i \end{pmatrix} = \frac{\hbar}{50}(12 + 12) = \frac{24}{50}\hbar$$

/10 8

$$e^{-i\hat{H}t/\hbar} \leadsto \begin{pmatrix} \cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}$$

Thus $|\Psi(t)\rangle \leadsto \begin{pmatrix} \cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} \end{pmatrix}$

2

$$|\Psi(t)\rangle = \cos\frac{\omega t}{2}\,|+\hat{z}\rangle + \sin\left(\frac{\omega t}{2}\right)|-\hat{z}\rangle$$

So $\text{Prob}\,(S_z = +\hbar/2) = |\langle +\hat{z}|\Psi(t)\rangle|^2$

2

$$= \cos^2\frac{\omega t}{2}$$

2

b) Describe the direction of the magnetic field so that it causes the evolution described above. Explain your answer.

The Hamiltonian has form

$$\hat{H} = \text{const}\ \hat{\vec{S}}\cdot\vec{B} = \text{const}\left[\hat{S}_x B_x + \hat{S}_y B_y + \hat{S}_z B_z\right]$$

Here we only have $\hat{S}_y$. So

$$\vec{B} = B_y\,\hat{y}$$

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Question 5

A particle with mass m is in an infinite square well with potential,

$$V(x) = \begin{cases} 0 & 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases}$$

The wavefunctions corresponding to normalized energy eigenstates, $|\phi_n\rangle$ for this system are

$$\phi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 \leqslant x \leqslant L \\ 0 & \text{otherwise.} \end{cases}$$

At a particular instant the normalized state of the particle, $|\Psi\rangle$, corresponds to

$$\Psi(x) = \begin{cases} A & \frac{L}{4} \leqslant x \leqslant \frac{3L}{4} \\ 0 & \text{otherwise} \end{cases}$$

where $A > 0$ is a constant.

2 a) Determine the constant A.

We need

$$\int_{-\infty}^{\infty} (\Psi(x))^2 \, dx = 1$$

$$\Rightarrow \int_{L/4}^{3L/4} A^2 dx = 1 \quad \Rightarrow \quad A^2 \, L/2 = 1$$

$$\Rightarrow \quad A = \sqrt{\frac{2}{L}}$$

Question 5 continued ...

Question 6

Do **either** part a) or part b) for full credit.

a) An ensemble of free particles with mass m are each in the state for which the normalized wavefunction is

$$\Psi(x) = \left(\frac{1}{\pi a^2}\right)^{1/4} e^{-x^2/2a^2}.$$

where $a>0$ has dimensions of length.

Determine the expectation value and the uncertainty of momentum measurements on this ensemble. Determine the expectation value of energy measurements on this ensemble.

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\, \hat{p}\, \Psi(x)\, dx$$

$$= \sqrt{\frac{1}{\pi a^2}} \int e^{-x^2/2a^2}\left(-i\hbar \frac{\partial}{\partial x}\right) e^{-x^2/2a^2} dx = -i\hbar \sqrt{\frac{1}{\pi a^2}} \int_{-\infty}^{\infty} x\, e^{-x^2/a^2} dx$$

$$= 0$$

$$\langle p^2 \rangle = \hbar^2 \int_{-\infty}^{\infty} \Psi^*(x) \left(-\frac{\partial^2}{\partial x^2}\right) \Psi(x)\, dx = -\hbar^2 \sqrt{\frac{1}{\pi a^2}} \int e^{-x^2/2a^2} \frac{\partial^2}{\partial x^2} e^{-x^2/2a^2}\, dx$$

$$= -\hbar^2 \sqrt{\frac{1}{\pi a^2}} \int_{-\infty}^{\infty} e^{-x^2/2a^2} \frac{\partial}{\partial x}\left[-\frac{x}{a^2} e^{-x^2/2a^2}\right] dx$$

$$= \hbar^2 \sqrt{\frac{1}{\pi a^2}} \frac{1}{a^2} \int_{-\infty}^{\infty} e^{-x^2/2a^2} \left[e^{-x^2/2a^2} - x\left(\frac{x}{a^2}\right) e^{-x^2/2a^2}\right] dx$$

$$= \hbar^2 \sqrt{\frac{1}{\pi a^2}} \left[\int_{-\infty}^{\infty} e^{-x^2/a^2} dx - 1/a^2 \int_{-\infty}^{\infty} x^2 e^{-x^2/a^2} dx\right] \frac{1}{a^2}$$

$$= \hbar^2 \sqrt{\frac{1}{\pi a^2}} \left[\sqrt{\pi a^2} - \frac{1}{a^2}\frac{1}{2}\sqrt{\pi}\left(\frac{1}{a^2}\right)^{-3/2}\right] \frac{1}{a^2} = \hbar^2/a^2$$

$$\Rightarrow \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{a}$$

Question 6 continued ...

Then $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\Rightarrow E = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar^2}{2ma^2}$$

Question 7

Do **either** part a) or part b) for full credit.

a) A particle in a spherically symmetric potential is in the state

$$|\psi\rangle = \frac{1}{2}\left[|1,1\rangle + \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right]$$

Determine $\hat{L}_x\,|\psi\rangle$ and use this to describe as precisely as possible (this could be statistical) what a measurement of L_x will yield for a particle in this state.

$$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$$

$$\Rightarrow \hat{L}_x|\psi\rangle = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)\left[|1,1\rangle + \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right]\frac{1}{2}$$

$$= \frac{1}{4}\Big\{\overset{0}{\cancel{L_+|1,1\rangle}} + \sqrt{2}\,\overbrace{L_+|1,0\rangle}^{\sqrt{2}|1,1\rangle} + \overbrace{L_+|1,-1\rangle}^{\sqrt{2}|1,0\rangle}$$

$$+ \underbrace{L_-|1,1\rangle}_{\sqrt{2}|1,0\rangle} + \sqrt{2}\,\underbrace{L_-|1,0\rangle}_{\sqrt{2}|1,-1\rangle} + \cancel{L_-|1,-1\rangle}\Big\}\hbar$$

$$= \frac{\hbar}{4}\left\{2\,|1,1\rangle + 2\sqrt{2}\,|1,0\rangle + 2\,|1,-1\rangle\right\}$$

$$= \hbar\,\frac{1}{2}\left[|1,1\rangle + \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right]$$

$$= \hbar\,|\psi\rangle$$

This is an eigenstate of $\hat{L}_x$ with eigenvalue $+\hbar$

$\Rightarrow$ measurement will yield $\underline{L_x = +\hbar}$ with $\underline{\text{certainty}}$

Question 7 continued ...