Quantum Computing with Ensembles

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Outline

"All information is physical."

- Classical computing and complexity.
- Quantum mechanics of qubits.
- Quantum computing: standard approaches.
- Quantum computing: ensemble approaches.

Classical mechanics governs the behavior and scope of conventional information processing devices (PCs, cell-phones, etc...).

Quantum mechanics extends information processing possibilities beyond those accessible to conventional classical information processing devices.

Integer Multiplication

How difficult is integer multiplication?

- Two single digit integers:

$$7 \times 8 = 56.$$

- Two two digit integers:

$$\begin{array}{r} 27 \\ 18 \\ \hline 216 \\ 27 \\ \hline 486 \end{array}$$

- Two three digit integers:

$$\begin{array}{r} 727 \\ 348 \\ \hline 5816 \\ 2908 \\ 2181 \\ \hline 252996 \end{array}$$

- Two n digit integers:

Approximately n^2 single digit multiplications and additions.

Polynomial Complexity

- Number of basic operations as function of input size n.
- Assess behavior for large n.
- Linear, $O(n)$:

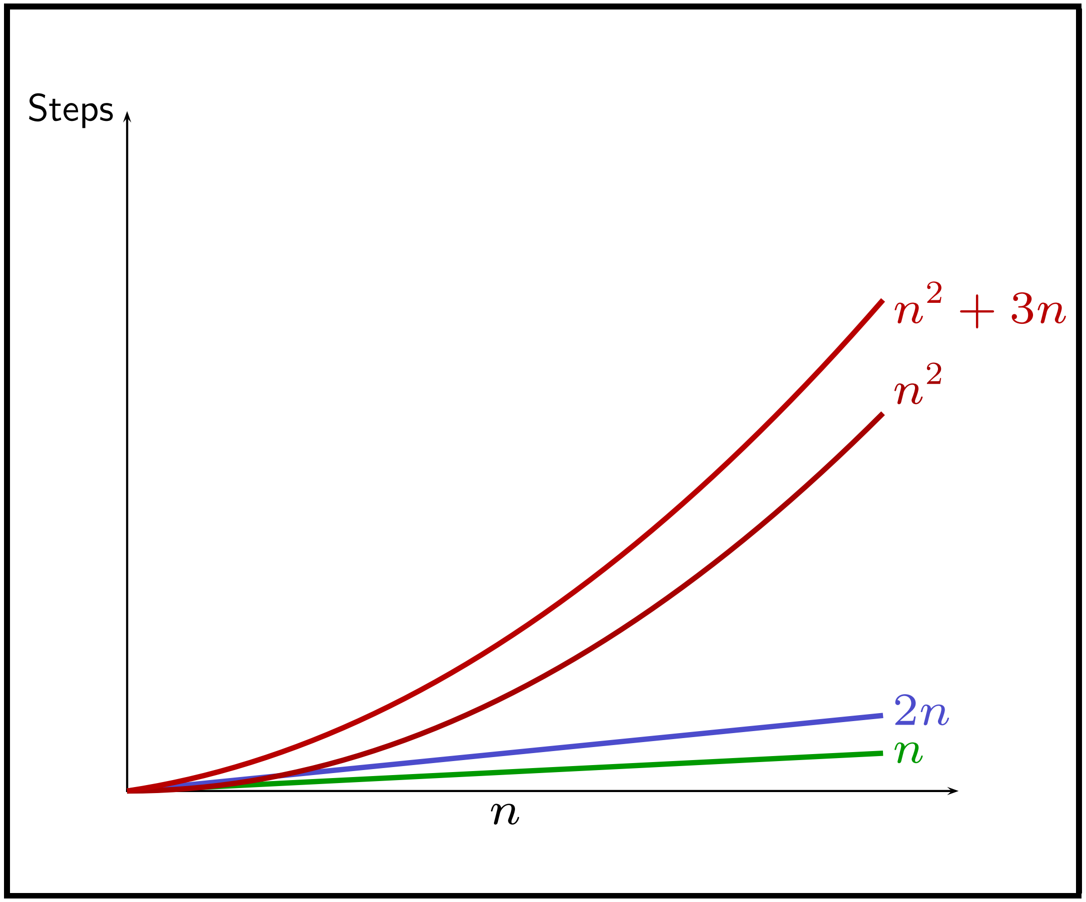
$$\text{Operations} = \alpha n + \ldots$$

- Quadratic, $O(n^2)$:

$$\text{Operations} = \alpha n^2 + \ldots$$

- Polynomial, $O(n^k)$:

$$\text{Operations} = \alpha n^k + \ldots$$



Integer Factorization

How difficult is integer factorization?

- Two digit integer:

$$91 = a \times b$$
$$\Rightarrow a = 7 \quad \text{and}$$
$$b = 13$$

- Three digit integer:

$$713 = a \times b$$
$$\Rightarrow a = ? \quad \text{and}$$
$$b = ?$$

- Trial and error factorization of n digit integer N. Number of guesses:

$$\sqrt{N} \simeq \sqrt{10^n} = 10^{n/2}$$ **Exponential in n.**

Best known integer factorization is exponential:
$$O((\exp(n^{1/3}(\log n)))^{2/3})$$

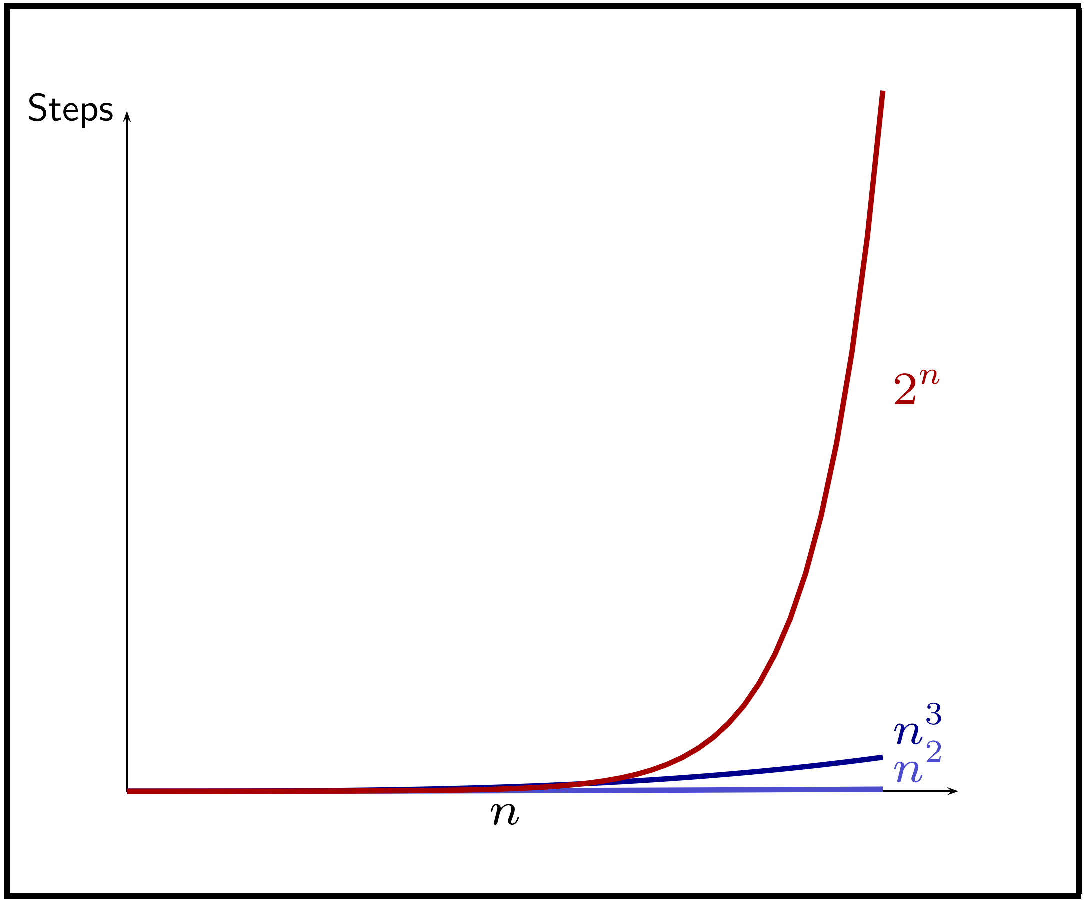
Computational Complexity

- How many additional digits to double the number of steps?
- Quadratic, $O(n^2)$:

$$n_{\text{new}} \simeq \sqrt{2}\, n_{\text{old}}$$

- Exponential, e.g. $O(2^n)$

$$n_{\text{new}} \simeq n_{\text{old}} + 1$$



Polynomial $\rightsquigarrow$ easy.
Exponential $\rightsquigarrow$ hard.

Classical Information Representation

Abstraction

- Binary digit (bit):

State is **one of** 0 or 1.

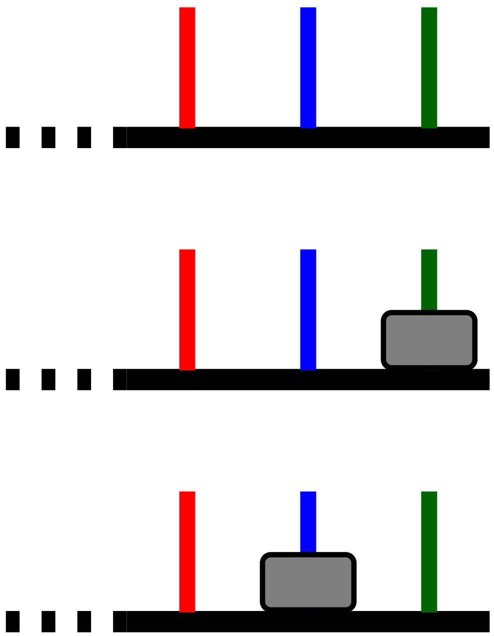
- Binary representation:

$$0 \equiv 000 \quad 1 \equiv 001 \quad 2 \equiv 010 \ldots$$

$$\text{PA} \equiv \underbrace{1001111}_{\text{P}}\ \underbrace{1000001}_{\text{A}}$$

Realization

- Pegs and beads



$\equiv 000$

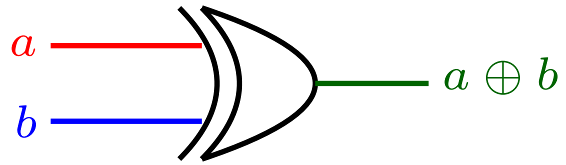
$\equiv 001$

$\equiv 010$

Classical Information Processing - Basic Gates

Abstraction

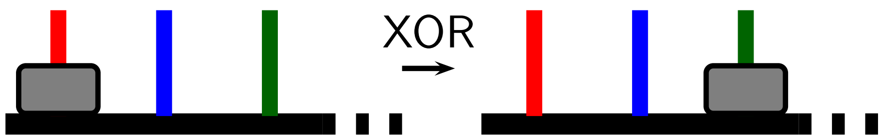
- Example: XOR on two bits



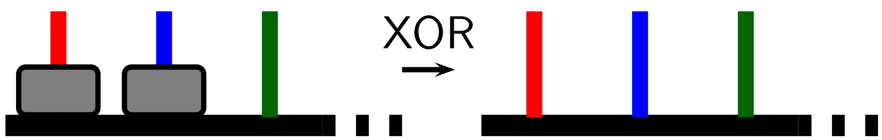
a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

Realization

- Example: XOR via pegs and beads
- Implementation rules:
  - Red and blue peg beads $\longrightarrow$ green peg.
  - Two beads on one peg $\longrightarrow$ remove both.
- 1 XOR 0



- 1 XOR 1



Classical information is usually viewed in the abstract.

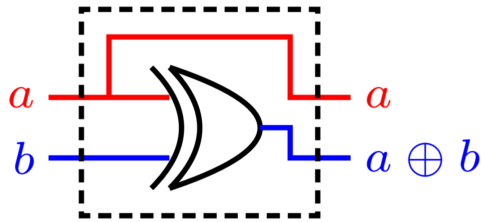
Reversible Computing

Reversible versions of basic gates exist.

- **Example:** Standard XOR:

  Find a, b if $a \oplus b = 0$

- Reversible XOR:



$$
\begin{array}{ccc}
0\ 0 & \longrightarrow & 0\ 0 \\
0\ 1 & \longrightarrow & 0\ 1 \\
1\ 0 & \longrightarrow & 1\ 1 \\
1\ 1 & \longrightarrow & 1\ 0
\end{array}
$$

- Vector representation for bit states:

$$
00 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad 01 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad 10 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad 11 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

- Matrix representation for gates:

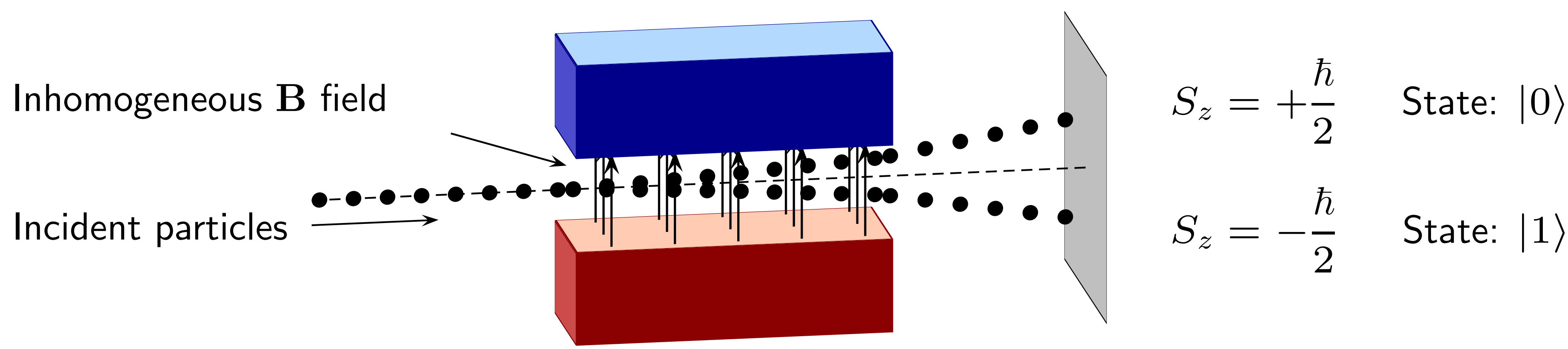
$$
\text{XOR} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

Quantum computing allows **superpositions of bit states.**

Spin $\frac{1}{2}$ Quantum Systems

Spin = intrinsic **angular momentum** of subatomic and atomic scale particles.

- **Stern-Gerlach** measures angular momentum via magnetic dipole moment.

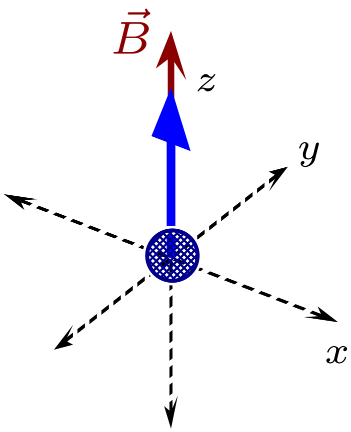


- Examples: electron, proton, H, ^{13}C.

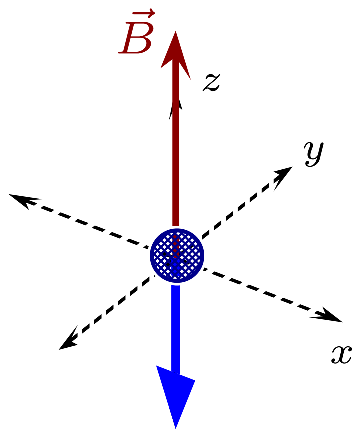
Quantum States and Information

Information is stored as a state a spin $\frac{1}{2}$ quantum system **(qubit).**

Energy Eigenstates



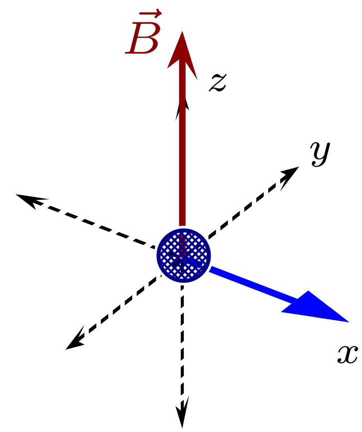
State: $|0\rangle$



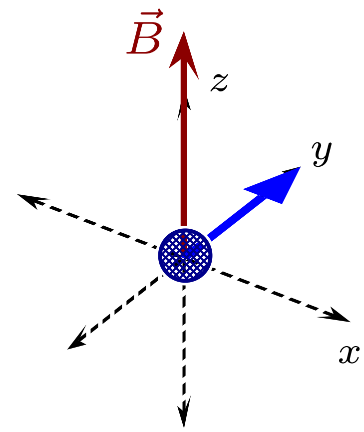
State: $|1\rangle$

Classical bit state.

Superposition states



State: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$



State: $\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$

Beyond classical bit states!

General state: $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \sim \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ where $|\alpha|^2 + |\beta|^2 = 1.$

Multiple Qubits

Multiple qubit states represented via tensor products.

- Two-qubit unentangled state (e.g. two spins along the x axis):

$$(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = |00\rangle + |01\rangle + |10\rangle + |11\rangle$$

where

$$|ab\rangle \equiv |a\rangle\,|b\rangle := |a\rangle \otimes |b\rangle\,.$$

- Quantum mechanics allows any (often entangled) superposition:

$$|\psi\rangle = \boldsymbol{\alpha}_0\,|00\rangle + \boldsymbol{\alpha}_1\,|01\rangle + \boldsymbol{\alpha}_2\,|10\rangle + \boldsymbol{\alpha}_3\,|11\rangle \sim \begin{pmatrix} \boldsymbol{\alpha}_0 \\ \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \end{pmatrix}$$

where

$$|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1\,.$$

Quantum Measurements and Information Extraction

Information is extracted via **quantum measurements**.

- Measurements of z component of single qubit spin are not deterministic:

$$\alpha\,|0\rangle + \beta\,|1\rangle \rightsquigarrow \begin{cases} S_z = +\hbar/2 \text{ with probability } |\alpha|^2 \\ S_z = -\hbar/2 \text{ with probability } |\beta|^2 \end{cases}$$

- Measurement induces "collapse" of state:

S_z	Bit value	State collapse
$+\hbar/2$	0	$\alpha\,\|0\rangle + \beta\,\|1\rangle \longrightarrow \|0\rangle$
$-\hbar/2$	1	$\alpha\,\|0\rangle + \beta\,\|1\rangle \longrightarrow \|1\rangle$

Quantum Dynamics and Information Processing

Information is processed via **unitary transformations ("gates")**.

- Linear time evolution

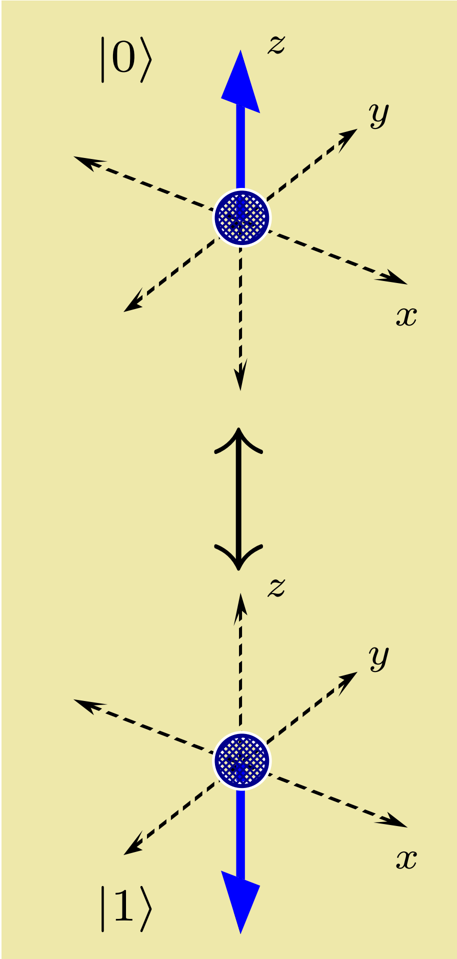
$$|\psi_{\text{final}}\rangle = \hat{U} \; |\psi_{\text{initial}}\rangle \quad \text{where} \quad \hat{U}^\dagger \hat{U} = \hat{I}.$$

- **Example:** Single qubit quantum NOT

$$\hat{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ transforms } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

and generalizes classical NOT

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

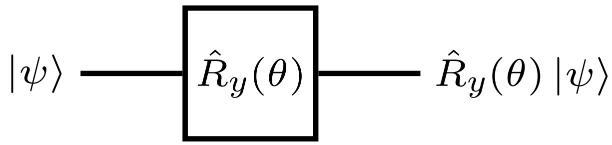


Quantum Gates

Reduction to one and two qubit unitary operations.

Single qubit rotations

- **Example:** Single bit rotation



$$\hat{R}_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

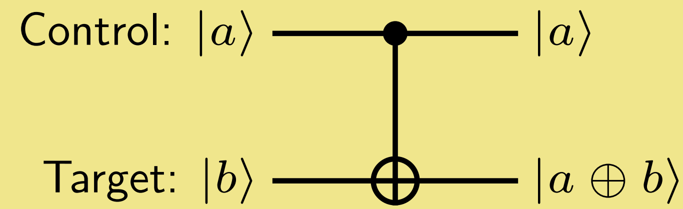
- On "classical" $|0\rangle$ (for $\theta = \pi/2$):

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Two-qubit gates

- **Example:** Controlled-NOT



$$\hat{U}_{\text{CN}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ on } \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Dynamics: Gate Construction

Gate construction via evolution under the system Hamiltonian.

- **Schrödinger equation:**

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

gives **unitary evolution**

$$|\psi(t)\rangle = \hat{U}(t, t_0)\ |\psi(t_0)\rangle$$

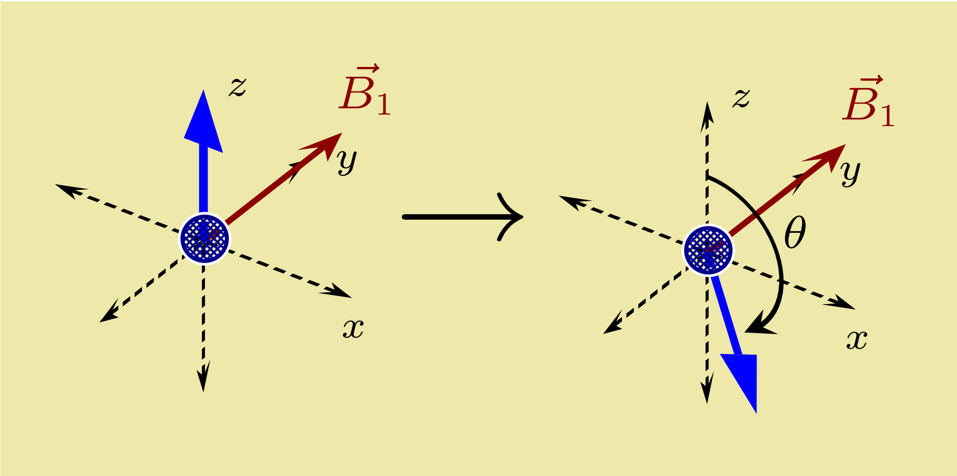
- For time independent $\hat{H}$:

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}.$$

- **Example:** Magnetic field along $\hat{y}$:

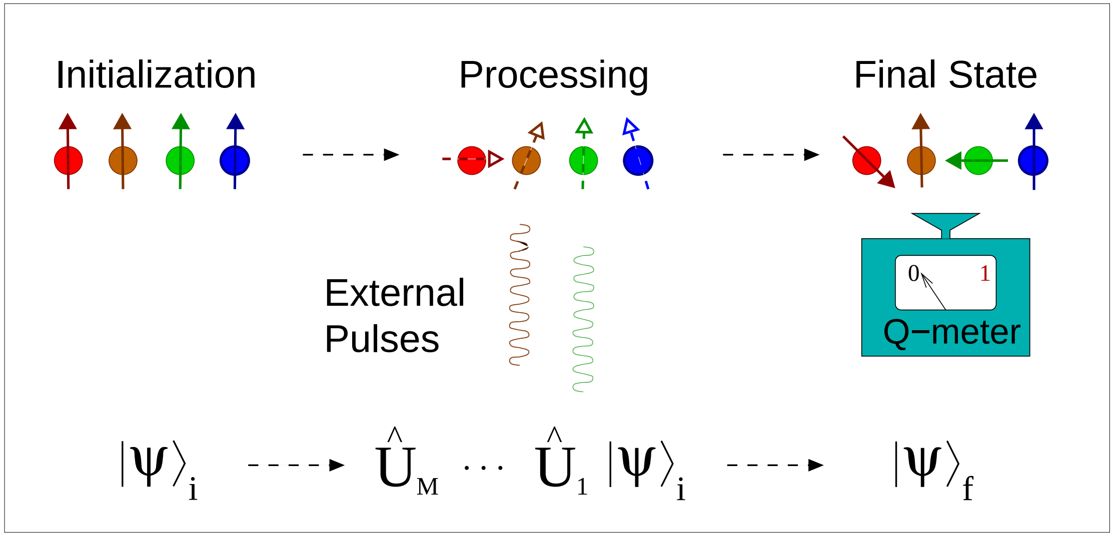
$$\hat{H} = \hbar\gamma B_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

applied for $t - t_0 = \frac{\theta}{2B_1\gamma}$



Quantum Computing Scheme

- Uses distinguishable qubits.



- Artful construction of evolution steps uses:
  - superpositions,
  - entangled states.

Quantum algorithms provide speedups **(fewer computational steps)**.

Quantum Algorithms

- **"Toy" algorithms:**
  - Global properties of binary functions.
  - Exponential speedup.
  - Deutsch-Jozsa, Bernstein-Vazirani and Simon's algorithms.

- **Searching (Grover):**
  - Search unstructured database.
  - Quadratic speedup in terms of oracle queries.

- **Integer factorization (Shor):**
  - Factorize integer $N = pq$.
  - Problem size $L := \log_2 N$.
  - Classical: $O(exp(L^{1/3}(\log L)))^{2/3}$.
  - Quantum: $O(L^3)$.

Decimal digits	Classical	Quantum
100	$\sim 10^{13}$	$\sim 10^{7}$
200	$\sim 10^{17}$	$\sim 10^{9}$
300	$\sim 10^{20}$	$\sim 10^{10}$
400	$\sim 10^{23}$	$\sim 10^{10}$

  - Age of universe $\sim 10^{17}$s.
  - Can break RSA code.

Deutsch-Jozsa Algorithm

Deutsch problem concerns properties of simple binary functions.

Single Bit Binary Functions

- Maps

$$\{0,1\} \xrightarrow{f} \{0,1\}$$

$$x \mapsto f(x) = ax \oplus b$$

where $a, b \in \{0,1\}$.

- Addition modulo 2:

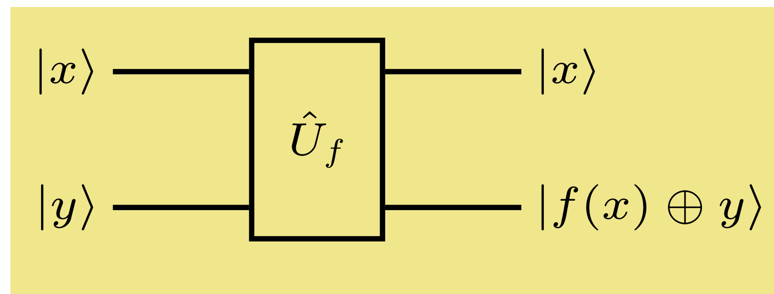
$$0 \oplus 0 := 0 \qquad 0 \oplus 1 := 1$$

$$1 \oplus 0 := 1 \qquad 1 \oplus 1 := 0$$

- Task: **Find a.**

Function Evaluation

- Use unitary function evaluation:



for $x, y \in \{0,1\}$.

- **"Classical"** approach requires **two function evaluations**:

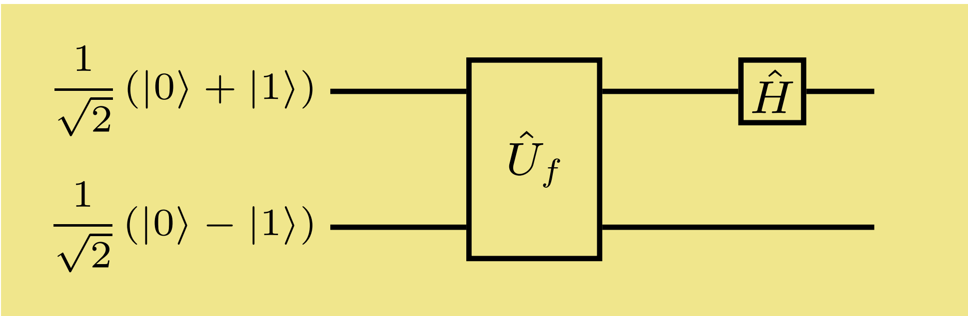
$$|0\rangle\,|0\rangle \quad \rightarrow \quad |0\rangle\,|b\rangle$$

$$|1\rangle\,|0\rangle \quad \rightarrow \quad |1\rangle\,|a \oplus b\rangle$$

Deutsch-Jozsa Algorithm

Quantum superposition helps to solve the Deutsch problem with **just one function evaluation!**

- Use **quantum superpositions.**



$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Upper qubit state **before Hadamard:**

If $\mathbf{a = 0}$: $$\frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $\mathbf{a = 1}$: $$\frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Upper qubit state **after Hadamard:**

If $\mathbf{a = 0}$: $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

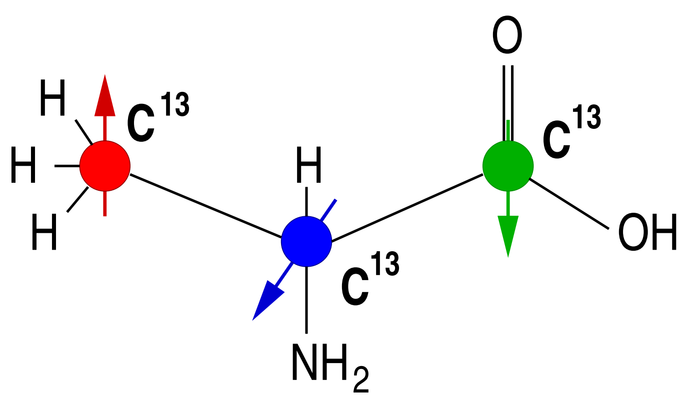
If $\mathbf{a = 1}$: $$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Spin z measurement yields a.

Nuclear Magnetic Resonance

Nuclear Spin Spectroscopy

- Spin $\frac{1}{2}$ nuclei in strong magnetic field, $\vec{B}_0$.
- Selective manipulation by tuning frequencies of external fields.
- Precession detected via readout coils.
- **Example:** H and ^{13}C nuclei of alanine.



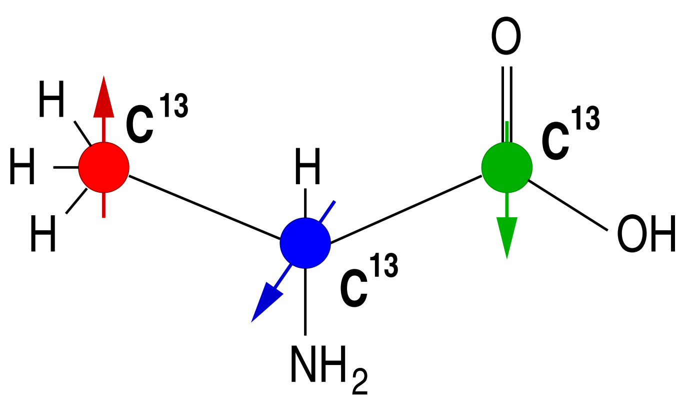
Source: Stoltz Group, Dept. of Chemistry, Caltech.

NMR Quantum Computing

NMR: an accessible technology for small scale quantum computers.

Qubits

- Distinct nuclear spins provide qubits.
- **Example:** The ^{13}C nuclei of alanine give three qubits



- Single qubit gates: spin selective external magnetic fields.
- Two qubit gates: evolution under spin-spin coupling.

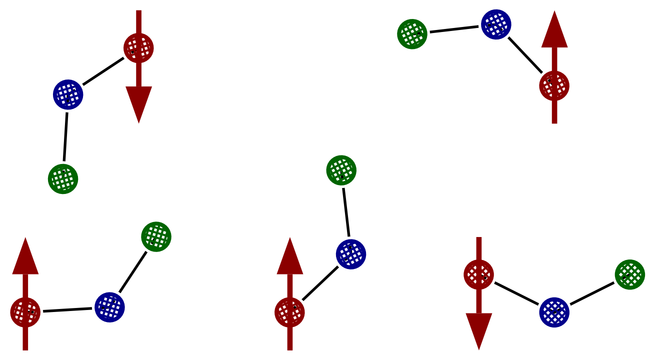
Issues

- Readily available technology.
- Weak interactions with environment $\Rightarrow$ many gates before information is degraded.
- Algorithm implementation:
  - Shor factorization - 7 qubits.
    Vandersypen et.al, Nature 414, 883-7 (20 Dec. 2001).
  - Grover search - 3 qubits.
    Vandersypen et.al, App. Phys. Lett. 76, 646-8 (2000).
- **Ensemble initialization and readout.**

Ensemble Quantum Computing

Ensemble of Identical Computers

- NMR sample with $\approx 10^{20}$ identical molecules.



- Rapid molecular motion $\Rightarrow$ no intermolecular interactions.

Statistically Mixed States

- Quantum state varies through ensemble, e.g. thermal equilibrium:

$$|0\rangle \text{ with prob } \approxeq \frac{1}{2}\left(1 + \frac{\hbar\omega}{2k_B T}\right)$$

$$|1\rangle \text{ with prob } \approxeq \frac{1}{2}\left(1 - \frac{\hbar\omega}{2k_B T}\right)$$

$\omega =$ precession frequency about $\vec{B}_0$.

- Weak polarization:

$$\hbar\omega/2k_B T \approx 10^{-4}$$

Mixed state input $\Rightarrow$ alternative initialization.
Ensemble average output $\Rightarrow$ alternative readout.

Ensembles: Initialization and Readout

Initialization

- Non-unitary scheme prepares pseudo-pure state.

Thermal eq. state
$\downarrow$
Completely random $+$ pure state

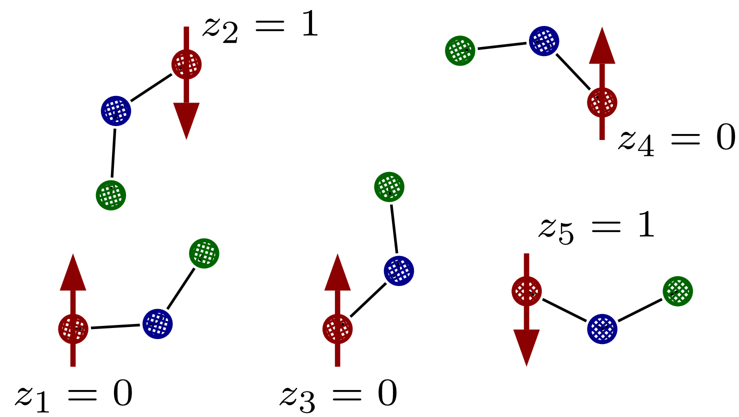
- Poor scaling common:

Signal strength $\sim n/2^n$

where $n =$ number of qubits.

Readout

- Sample averages over ensemble with M members.



$$\overline{z} = \sum z_i/M$$

- Majority vote decisions: $\overline{z} \overset{?}{>} 1/2$.

Non deterministic output.

Modified Algorithms for Ensemble QC

Readout

- Converted "deterministic" algorithms
  - Grover search (one marked item) - unnecessary.
  - Grover search (few marked items) - few runs plus filtering.
  - Shor factorization - duplication of quantum computers.
- Modified algorithms require fewer steps:

Grover search can be truncated.

D. Collins, Phys. Rev. A 65, 052321 (2002).

Initialization

- Use noisy thermal equilibrium input states?
- **Bernstein-Vazirani algorithm:**
  - Standard thermal equilibrium state plus unmodified algorithm plus expectation values satisfactory.
- **Deutsch-Jozsa algorithm:**
  - Existing "one pure qubit plus maximally mixed state" approach unsatisfactory.

    Arvind, D. Collins, Phys. Rev. A 68, 052301 (2003).
- **Grover, Shor:** - ?

Single Bit Output: Statistics

Framework

- Pure state quantum algorithm:

$$|\psi_i\rangle \rightarrow |\phi_z\rangle\,|z\rangle$$

  where $z = 0, 1 \rightsquigarrow$ algorithm output.

- Mixed state algorithm initial state:

$$\rho_i = \frac{1-\varepsilon}{2^n}\hat{I}^{\otimes n} + \varepsilon\,|\psi_i\rangle\,\langle\psi_i|$$

- Polarization $\varepsilon \sim$ fraction of molecules in pure state gives:

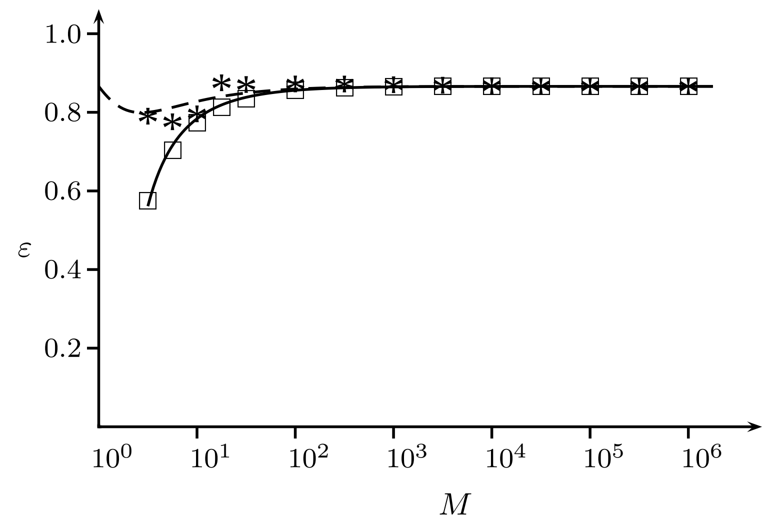
$$\Pr(z) = \frac{1+\varepsilon}{2}$$
$$\Pr(1-z) = \frac{1-\varepsilon}{2}$$

Classical vs Quantum Ensemble

- For given ensemble size, M:

Polarization required for quantum to outperform classical probabilistic using comparable resources?

- Deutsch-Jozsa



B Anderson, D. Collins, Phys. Rev. A 72, 042337 (2005).

Quantum Information Arena

Theory

- Quantum Cryptography
- Quantum Teleportation
- Superdense Coding
- Decoherence and Error Correction
- Entanglement
- Quantum Channels

Practice

- NMR
- Photons
- Trapped Ions
- Quantum Dots
- Doped Silicon
- Superconducting Circuits

Future Directions

Ensemble QC

- Can other standard quantum algorithms and applications be tailored for ensemble QC?
- What quantum resources does ensemble QC require?
- Where does ensemble QC lie in relation to standard QC and classical computation?

General

- Quantum mechanics provides a new information processing paradigm.
- Information is stored and manipulated in ways fundamentally different from those for classical information processing.
- The laws of physics dictate information processing possibilities and limitations.