Discrimination of Unitary Transformations and Quantum Algorithms

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Abstract

Quantum algorithms are typically understood in terms of the evolution of a multi-qubit quantum system under a prescribed sequence of unitary transformations. In this picture, the input to the algorithm prescribes some of the unitary transformations in the sequence with others remaining fixed. In the context of oracle query algorithms, the input determines the oracle unitary transformation. Such algorithms can be regarded as devices for discriminating amongst a set of unitary transformations. The question arises: "Given a set of known oracle unitary transformations, to what extent is it possible to discriminate amongst them?"

We investigate this for the Deutsch-Jozsa algorithm. The task of discriminating amongst the admissible oracle unitary transformations for the Deutsch-Jozsa problem is shown to result in an exhaustive collection of algorithms which can solve the problem with certainty.

Oracle Algorithms as Unitary Discriminators

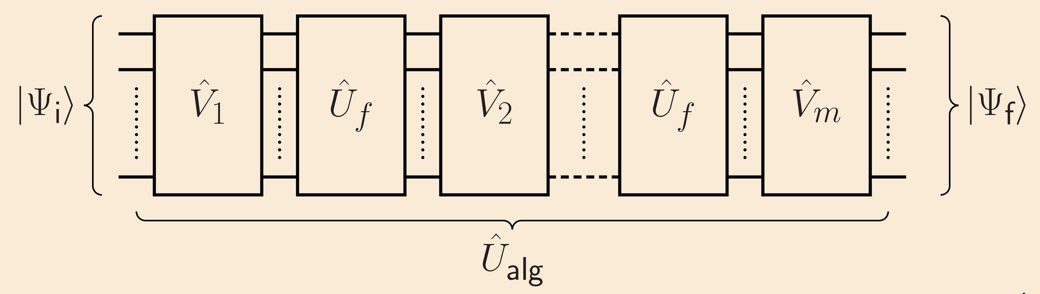
Binary **oracle functions** appear in certain quantum algorithms:

$$f : \{0,1\}^n \mapsto \{0,1\}.$$

One possible implementation via a unitary transformation, used in the **Deutsch-Jozsa and Grover's algorithms,** [1, 2] requires n qubits and acts on computational basis elements as:

$$\hat{U}_f \left|x_n \ldots x_1\right\rangle := (-1)^{f(x)} \left|x_n \ldots x_1\right\rangle.$$

The oracle is interspersed with oracle independent unitaries, $\hat{V}_1 \ldots \hat{V}_m$, during the algorithm.



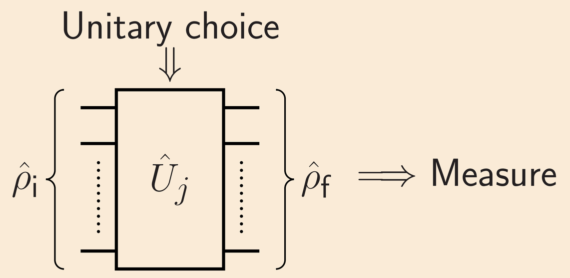
- Algorithm **input** ⇝ an admissible oracle function ⇝ **algorithm unitary,** $\hat{U}_{alg}$.
- Algorithm **output** ⇝ determines oracle function/class ⇝ **determines algorithm unitary.**

The quantum algorithm is a means for discriminating between classes of admissible oracle unitary transformations.

Discrimination between Unitary Transformations

Unitary discrimination, based on quantum state discrimination [3, 4] entails:

- A collection of known unitaries, $\{\hat{U}_1, \hat{U}_2 \ldots\}$, and *a priori* probabilities $\{p_1, p_2 \ldots\}$,
- an input state, described via a density operator, ρ_i, which is the same for all admissible unitaries,
- a measurement described via a POVM, with elements $\hat{E}_1, \hat{E}_2, \ldots$, and applied to $\hat{\rho}_f$,
- an inference from the measurement outcome, e.g. attain the measurement outcome associated with $E_j \Rightarrow \hat{U}_j$ applied.



The problem of discriminating between the unitaries is equivalent to that of discriminating between the possible quantum states that they produce, i.e. between:

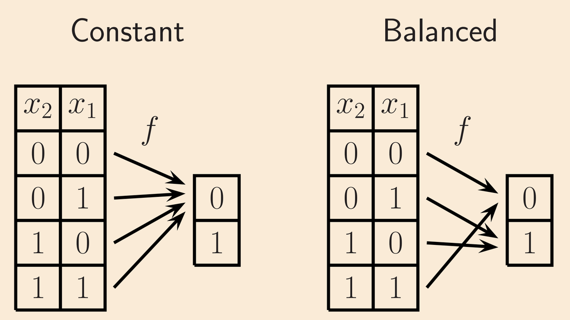
$$\hat{\rho}_{f\,j} := \hat{U}_j \hat{\rho}_i \hat{U}_j^\dagger.$$

Deutsch-Jozsa Problem

The Deutsch-Jozsa problem considers binary functions on n binary arguments

$$f : \{0,1\}^n \mapsto \{0,1\}$$

each of which is required to be in one of **two classes: constant or balanced.** For a balanced function, f returns 0 over exactly half of the possible arguments and 1 over the remaining half.



The problem is to **determine the function class** with a minimum number of oracle invocations.

The classical algorithm which determines function class **with certainty requires $2^n/2+1$ oracle invocations.**

The Deutsch-Jozsa algorithm [5], in its modified form [1], uses n qubits and the oracle

$$\hat{U}_f \left|x\right\rangle := (-1)^{f(x)} \left|x\right\rangle$$

where $\left|x\right\rangle := \left|x_n \ldots x_1\right\rangle$. The question that we address is:

Could we arrive at quantum algorithms that determine function class with certainty by considering distinguishability of the oracle unitaries for the two classes of functions, using just one oracle invocation?

Quantum Operations in the Deutsch-Jozsa Problem

Assuming that **the a priori probabilities for all balanced functions are the same,** it can be shown that the problem is equivalent to that of discriminating between the quantum operations:

constant functions: $\hat{\rho}_i \mapsto \hat{\rho}_{f\,const} = \hat{\rho}_i$

balanced functions: $\hat{\rho}_i \mapsto \hat{\rho}_{f\,bal} = \dfrac{1}{N-1}\left\{-\hat{\rho}_i + N\sum_{x=0}^{N-1} \hat{P}_x \hat{\rho}_i \hat{P}_x\right\}$

where $N = 2^n$ and

$$\hat{P}_x := \left|x\right\rangle\left\langle x\right|.$$

Discriminating between the two operations is equivalent to discriminating between the two states $\hat{\rho}_{f\,const}$ and $\hat{\rho}_{f\,bal}$.

Perfect Discrimination between Two States

For **conclusive discrimination** between two states $\hat{\rho}_1$ and $\hat{\rho}_2$, the POVM has two elements: $\hat{E}_1$ and $\hat{E}_2 = \hat{I} - \hat{E}_1$. If the measurement outcome attained is that associated with $\hat{E}_j$ the inference is that the state prior to measurement was $\hat{\rho}_j$. The probability that the inference is correct is

$$\Pr(\text{correct inference}) = p_1 \,\mathrm{Tr}\left(\hat{\rho}_1 \hat{E}_1\right) + p_2 \,\mathrm{Tr}\left(\hat{\rho}_2 \hat{E}_2\right).$$

For **correct inference with certainty for all a priori probabilities,** $\mathrm{Tr}\left(\hat{\rho}_j \hat{E}_j\right) = 1$ for $j = 1, 2$. It can be shown that this is implies:

- E_j is the identity operator on the support (subspace orthogonal to the kernel) of $\hat{\rho}_j$.
- The supports of $\hat{\rho}_1$ and $\hat{\rho}_2$ are orthogonal, i.e. $\hat{\rho}_1\hat{\rho}_2 = \hat{\rho}_2\hat{\rho}_1 = 0$.

Algorithms for Solving the Deutsch-Jozsa Problem

The requirement for perfect discrimination between constant and balanced operations yields

$$\left[\hat{\rho}_i, \sum_{x=0}^{N-1} \hat{P}_x \hat{\rho}_i \hat{P}_x\right] = 0 \quad \text{and}$$
$$N \sum_{x=0}^{N-1} \hat{P}_x \hat{\rho}_i \hat{P}_x \hat{\rho}_i = \hat{\rho}_i^2.$$

The diagonal form of the density operator is

$$\hat{\rho}_i = \sum_{j=1}^{M} p_j \left|\phi_j\right\rangle\left\langle\phi_j\right|$$

where $\{\left|\phi_j\right\rangle\}$ are orthonormal and M is the number of non-zero eigenvalues of $\hat{\rho}_i$. It can be shown that the previous conditions on $\hat{\rho}_i$ imply that the state is a pure state, i.e. $M=1$ and for this state

$$\sum_{x=0}^{N-1} \left|\left\langle\phi_1|x\right\rangle\right|^4 = \frac{1}{N}.$$

This together with the normalization requirement for $\left|\phi_1\right\rangle$, these implies that the initial state which will successfully discriminate between the two function classes is:

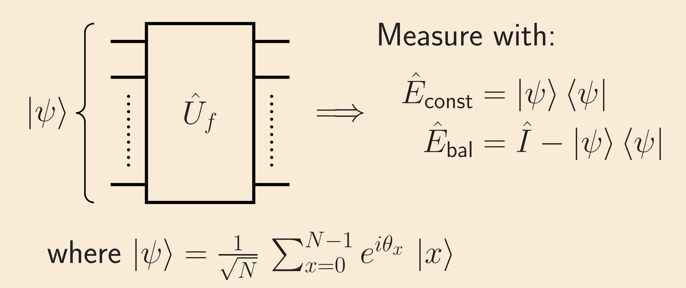
$$\hat{\rho}_i = \left|\phi_1\right\rangle\left\langle\phi_1\right| \quad \text{where}$$
$$\left|\phi_1\right\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{i\theta_x} \left|x\right\rangle$$

and θ_x are arbitrary real phases. The POVM elements corresponding to the function class outcomes are:

$$\hat{E}_{const} = \left|\phi_1\right\rangle\left\langle\phi_1\right| \quad \text{and}$$
$$\hat{E}_{bal} = \hat{I} - \left|\phi_1\right\rangle\left\langle\phi_1\right|.$$

Conclusions

The only algorithms which can solve the Deutsch-Jozsa problem with certainty for arbitrary a prior probabilities of function class choice are as illustrated.



Can the success probability of the Deutsch-Jozsa algorithm starting with specific classes of mixed states be assessed by considering discrimination of unitaries?

References

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