Optimal Estimation of Single Qubit Quantum Evolution Parameters

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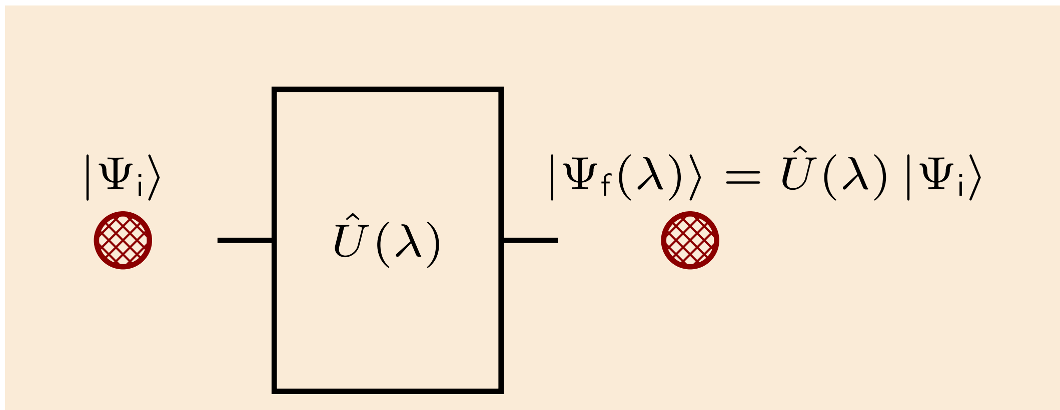
Parameter-dependent Unitary Evolution.

Quantum system evolution is described via unitary operations.



Task: Knowing nature of evolution, estimate parameters as accurately as possible by subjecting quantum system to evolution.

- Mixed states:

$$\hat{\rho}_{\mathsf{f}}(\lambda) = \hat{U}(\lambda)\ \hat{\rho}_{\mathsf{i}}\ \hat{U}(\lambda)^{\dagger}$$

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Statistical Nature of Measurement Outcomes and Estimation

Quantum Measurements

Measurements described in terms of POVMs.

- Possible measurement outcomes:

$$\{m_1, m_2, m_3, \ldots\}$$

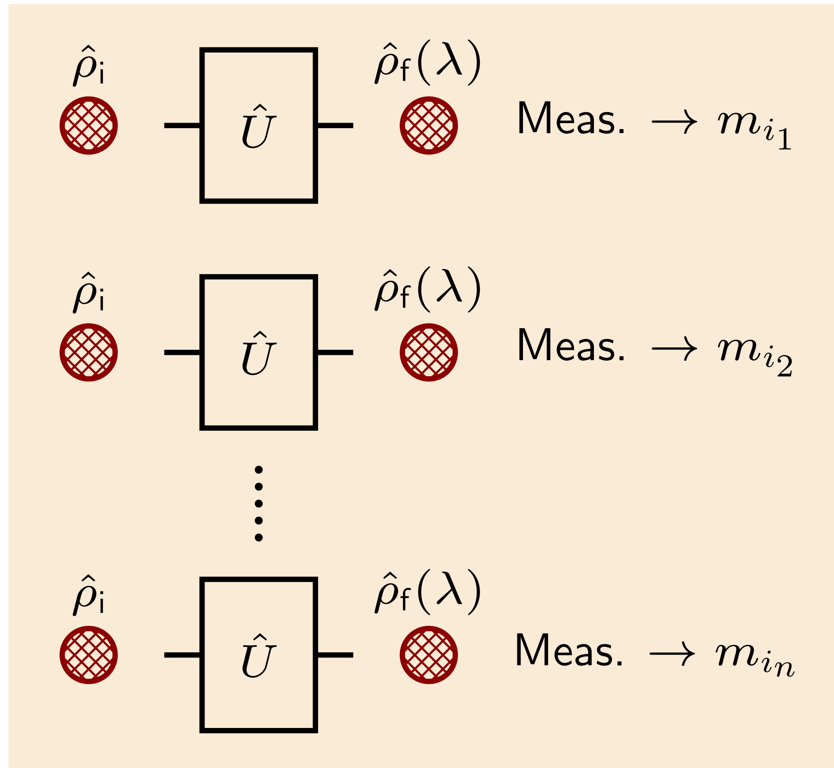
- Associated POVM elements:

$$\left\{\hat{\Pi}_1, \hat{\Pi}_2, \hat{\Pi}_3, \ldots\right\}$$

- Probabillities:

$$\Pr(m_j) = \mathrm{Tr}\left(\hat{\Pi}_j \hat{\rho}_{\mathrm{f}}(\lambda)\right)$$

Estimation via repeated measurement.



- Estimator for λ:

$$\lambda_{\mathrm{est}} = \lambda_{\mathrm{est}}(m_{i_1}, m_{i_2}, \ldots, m_{i_n}).$$

Classical Parameter Estimation

Estimator is probabilistic - distribution determined by measurement outcome distribution.

- Accuracy assessed via mean square error

$$\mathrm{m.\,s.\,e}\left(\lambda_{\text{est}}\right) = \left\langle \left(\lambda_{\text{est}} - \lambda\right)^2 \right\rangle .$$

- Unbiased estimator: $\langle \lambda_{\text{est}} \rangle = \lambda$ gives **classical Cramér-Rao bound:**

$$\mathrm{m.\,s.\,e}\left(\lambda_{\text{est}}\right) = \mathrm{var}\left(\lambda_{\text{est}}\right) \geqslant \frac{1}{F(\lambda)}$$

where **Fisher information is independent of estimator** and determined from probability distribution for measurement outcomes by

$$F(\lambda) = \left\langle \left(\frac{\partial \ln P(m_{i_1}, \ldots, m_{i_n} | \lambda)}{\partial \lambda} \right)^2 \right\rangle$$

Quantum Parameter Estimation

- Fisher information depends on system state, $\hat{\rho}_{\mathsf{f}}(\lambda)$, and **choice of measurement type** via

$$P(m_{i_1},\ldots,m_{i_n}|\lambda) = \mathrm{Tr}\left(\hat{\Pi}_{i_1}\hat{\rho}_{\mathsf{f}}\right)\ldots\mathrm{Tr}\left(\hat{\Pi}_{i_n}\hat{\rho}_{\mathsf{f}}\right).$$

- Quantum Fisher information gives bound for **any conceivable measurement**

$$F(\lambda) \leqslant H(\lambda)$$

where quantum Fisher information depends on system state via

$$H(\lambda) = \mathrm{Tr}\left(\hat{\rho}_{\mathsf{f}}\hat{L}^2\right)$$

with score operator $\hat{L}$ satisfying

$$\frac{\partial\hat{\rho}_{\mathsf{f}}}{\partial\lambda} = \frac{1}{2}\left(\hat{\rho}_{\mathsf{f}}\hat{L} + \hat{L}\hat{\rho}_{\mathsf{f}}\right)$$

Optimal estimation: choose input state so that $\hat{\rho}_{\mathsf{f}}$ **maximizes quantum Fisher information**.

General Unitary Parameter Estimation

Single parameter unitary

- Unitary on single system

$$\hat{U} = e^{-i\hat{G}\lambda/2}$$

with $\hat{G}$ Hermitian.

- Convexity implies pure input state optimal *Fujiwara, et.al. PRA **63** 042304 (2001)*. Pure output state

$$\hat{\rho}_{\mathsf{f}} = |\psi_{\mathsf{f}}(\lambda)\rangle \langle\psi_{\mathsf{f}}(\lambda)|$$

with

$$|\psi_{\mathsf{f}}(\lambda)\rangle = \hat{U}(\lambda)\,|\psi_{\mathsf{i}}\rangle \,.$$

Fisher Information

- For pure output state, Fisher information

$$H(\lambda) = 4\left[\frac{\partial \langle\psi|}{\partial\lambda}\frac{\partial |\psi\rangle}{\partial\lambda} + \left(\langle\psi|\,\frac{\partial|\psi\rangle}{\partial\lambda}\right)^2\right]$$

- Fisher information via generator:

$$H(\lambda) = \left[\langle\psi_{\mathsf{i}}|\,\hat{G}^2\,|\psi_{\mathsf{i}}\rangle - \left(\langle\psi_{\mathsf{i}}|\,\hat{G}\,|\psi_{\mathsf{i}}\rangle\right)^2\right].$$

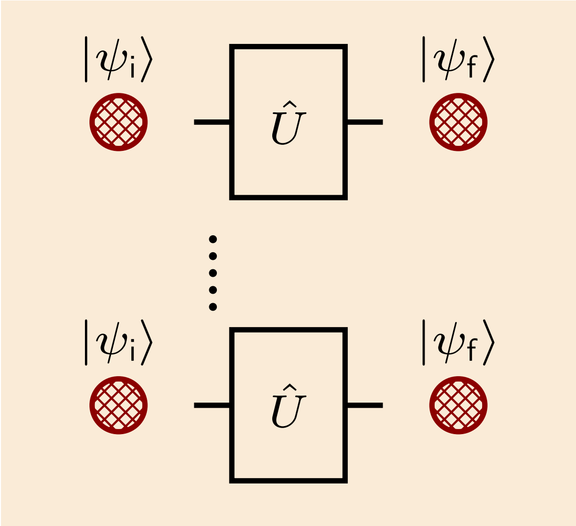
- Fisher information via extreme generator eigenvalues (*Giovanetti, et.al. PRL **96** 010401 (2006)*):

$$H(\lambda) = g_{\mathsf{max}} - g_{\mathsf{min}}$$

Single Qubit Unitary Parameter Estimation

Unentangled Input

- Identical state for each qubit:

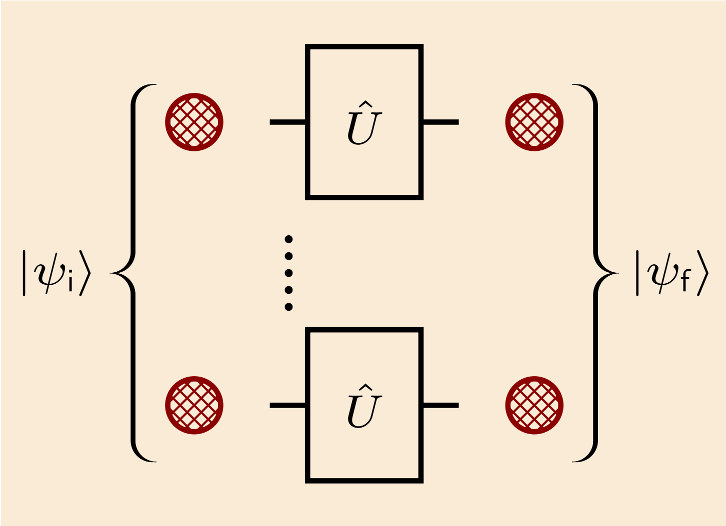


- Optimal input state gives

$$H(\lambda) = n \quad \Rightarrow \quad \mathrm{var}\left(\lambda_{\mathrm{est}}\right) \geqslant \frac{1}{n}$$

Entangled Input

- Entangled state, no ancilla:



- Optimal input state gives

$$H(\lambda) = n^2 \quad \Rightarrow \quad \mathrm{var}\left(\lambda_{\mathrm{est}}\right) \geqslant \frac{1}{n^2}$$

Non-Unitary Parameter Estimation

Evolution of quantum system in presence of other external quantum systems described in terms of quantum operations.

Quantum Channel and Operation

- Density operator evolution

$$\hat{\rho}_{\mathsf{f}}(\lambda) = \sum_j \hat{E}_j(\lambda)\hat{\rho}_{\mathsf{i}}\hat{E}_j^\dagger(\lambda)$$

where Kraus operators satisfy $\sum_j \hat{E}_j^\dagger \hat{E}_j \leqslant \hat{I}$.

Estimate λ with minimum number of channel uses.

Example: Bit-Flip Channel

- Single qubit evolution:

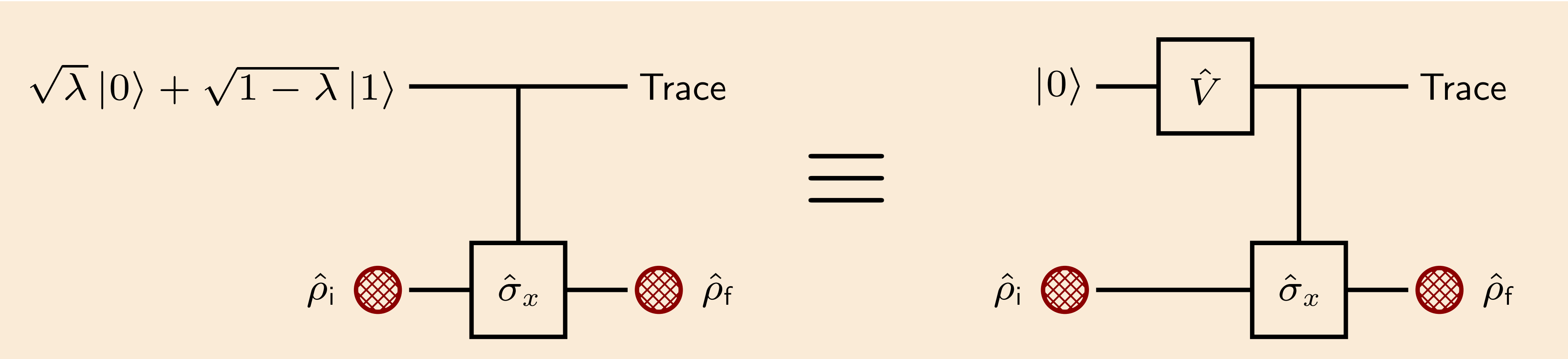
Do nothing with probability λ, flip qubit with probability $1-\lambda$.

- Kraus Operators:

$$\hat{E}_0 = \sqrt{\lambda}\ \hat{I}$$

$$\hat{E}_1 = \sqrt{1-\lambda}\ \hat{\sigma}_x$$

Bit-Flip Parameter Estimation



Quantum Circuit

- Parameter via:

$$\hat{V}(\lambda) = \begin{pmatrix} \sqrt{\lambda} & -\sqrt{1-\lambda} \\ \sqrt{1-\lambda} & \sqrt{\lambda} \end{pmatrix}$$

- Ancilla qubit constrained to input of $|0\rangle$.

Optimal Parameter Estimation

- Any measurements (both qubits) gives maximum Fisher information which only depends on $\hat{V}$. Optimal channel Fisher information satsifies:

$$H_{\mathsf{channel}}(\lambda) \leqslant H_{\mathsf{both}}(\lambda) = \frac{1}{\lambda(1-\lambda)}$$

Optimal Bit-Flip Parameter Estimation

Multiple Channel Uses

- Ancilla bits are constrained to $|0\rangle$.

Ancilla bits are unentangled.

- Channel bits are possibly entangled.
- Any measurement on all qubits $\Rightarrow$ channel qubits irrelevant and optimal Fisher information is

$$H_{\text{both}}(\lambda) = \frac{n}{\lambda(1-\lambda)}.$$

- Optimal channel Fisher information bounds:

$$H_{\text{channel}}(\lambda) \leqslant H_{\text{both}}(\lambda) = \frac{n}{\lambda(1-\lambda)}$$

Channel Fisher Information

- For single channel use with channel pure input state

$$\hat{\rho}_{\text{i}} = \frac{1}{2}\left[\hat{I} + \hat{\sigma}_y\right],$$

Fisher information is

$$H_{\text{channel}}(\lambda) = \frac{1}{\lambda(1-\lambda)}$$

- For n unentangled uses

$$H_{\text{channel}}(\lambda) = \frac{n}{\lambda(1-\lambda)}$$

Entangled channel states give no advantage.

Conclusions

- Optimal Fisher information for n uses of single qubit unitary with **unentangled inputs**:

$$H(\lambda) = n$$

- Optimal Fisher information for n uses of single qubit unitary with **entangled inputs**:

$$H(\lambda) = n^2$$

- Optimal Fisher information for n uses of single qubit bit flip with **unentangled inputs**:

$$H(\lambda) = \frac{n}{\lambda(1-\lambda)}$$

Entangled input states offer quadratic advantage for unitary estimation. Entangled channel states give no advantage for bit-flip estimation.