No Advantage to Entanglement in Bit Flip Parameter Estimation

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Abstract

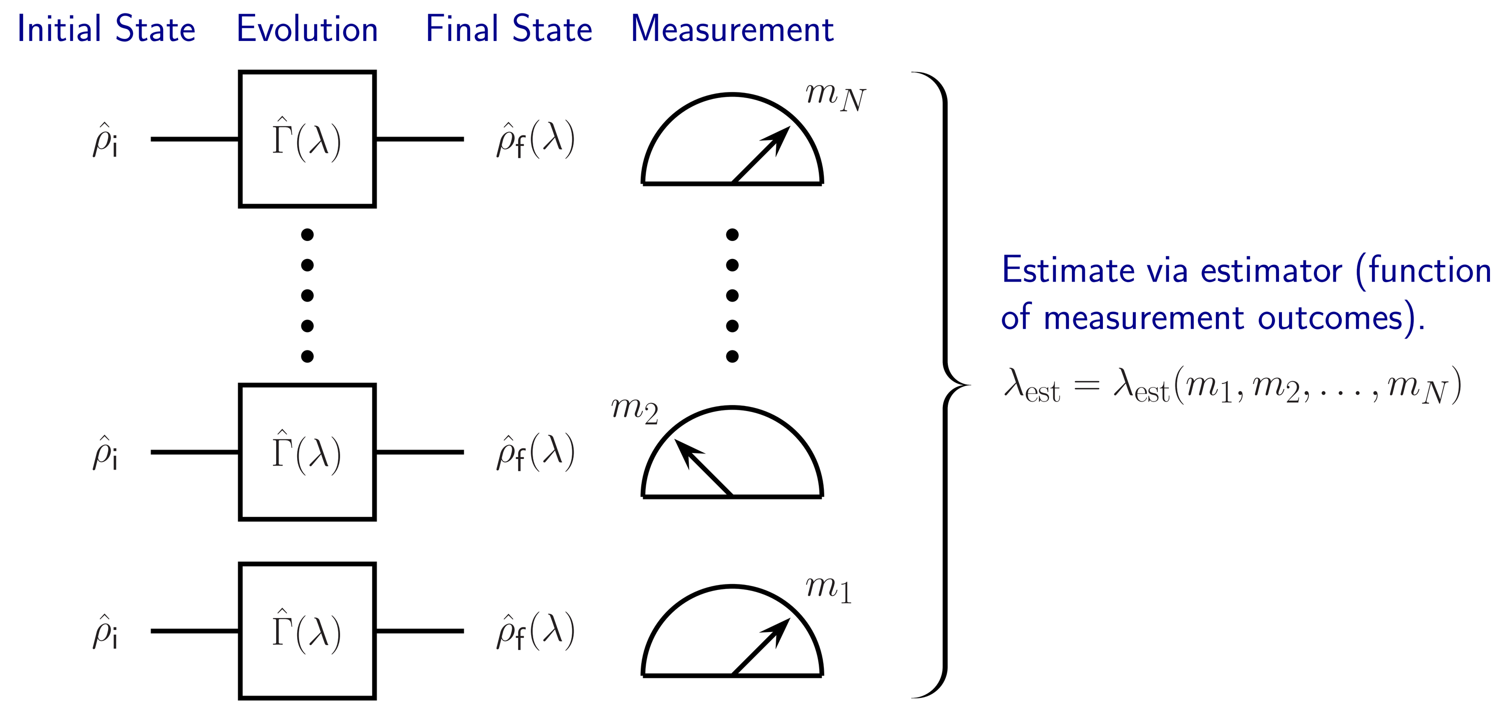
We consider optimal estimation of the parameter describing a bit-flip channel. Using the quantum Fisher information as a measure of the accuracy of the parameter estimation, we show that entanglement offers no advantage for multiple uses of the channel. This contrasts with parameter estimation in depolarizing channels, where entanglement offers a modest advantage and unitary channels where entanglement offers a distinct advantage.

The evolution of any quantum system is described by a quantum operation, which may depend on one or more parameters. Examples of parameter dependent evolution:

- optical phase shift in interferometry and
- depolarizing quantum channels.

Task: Knowing the type of evolution, estimate parameters as accurately as possible by subjecting quantum systems to the evolution.

The probabilistic nature of outcomes of measurements on quantum systems and the effects of measurements on quantum states imply that the quantum operation, $\hat{\Gamma}(\lambda)$, must be invoked repeatedly (N times).



Parameter estimates fluctuate statistically between repeated runs, each with N quantum operation invocations on the same input state. The statistics of the outcomes of measurements on the quantum systems determine the probabilities of various estimates.

Cramér-Rao bound and Fisher Information

The accuracy of the measurement is quantified in terms of the mean square error,

$$\text{m.s.e}\left(\lambda_{\text{est}}\right) = \left\langle \left(\lambda_{\text{est}} - \lambda\right)^2 \right\rangle.$$

For any unbiased estimator, the **Cramér-Rao bound** gives [1]:

$$\text{m.s.e}\left(\lambda_{\text{est}}\right) \geqslant \frac{1}{F(\lambda)} \geqslant \frac{1}{H(\lambda)}.$$

MSE depends on estimator choice.

Quantum Fisher information depends on initial state but not measurement.

Classical Fisher information depends on measurement choice but not estimator.

The quantum Fisher information is given by

$$H(\lambda) = \text{Tr}\left(\hat{\rho}_{\text{f}} \hat{L}^2\right) \qquad \text{with} \qquad \frac{\partial \hat{\rho}_{\text{f}}}{\partial \lambda} = \frac{1}{2}\left(\hat{\rho}_{\text{f}} \hat{L} + \hat{L} \hat{\rho}_{\text{f}}\right)$$

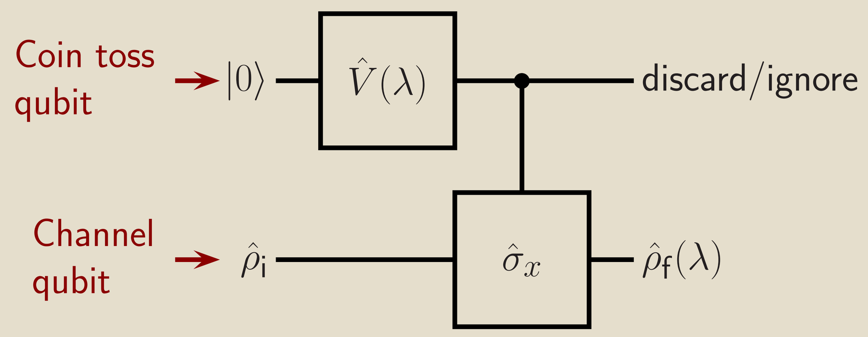
Optimal estimation: choose input state and additional parameter-independent unitaries so as to maximize the quantum Fisher information.

Parameter Estimation for the Bit Flip Channel

A bit flip channel acts on a single qubit via

$$\hat{\rho}_{\text{i}} \mapsto \hat{\rho}_{\text{f}}(\lambda) = \lambda \hat{\rho}_{\text{i}} + (1-\lambda)\hat{\sigma}_x \hat{\rho}_{\text{i}} \hat{\sigma}_x.$$

The bit flip can be attained via unitary evolution with an additional "coin toss" qubit. This is an example of a **programmable channel** [2].



The "coin flip" unitary is

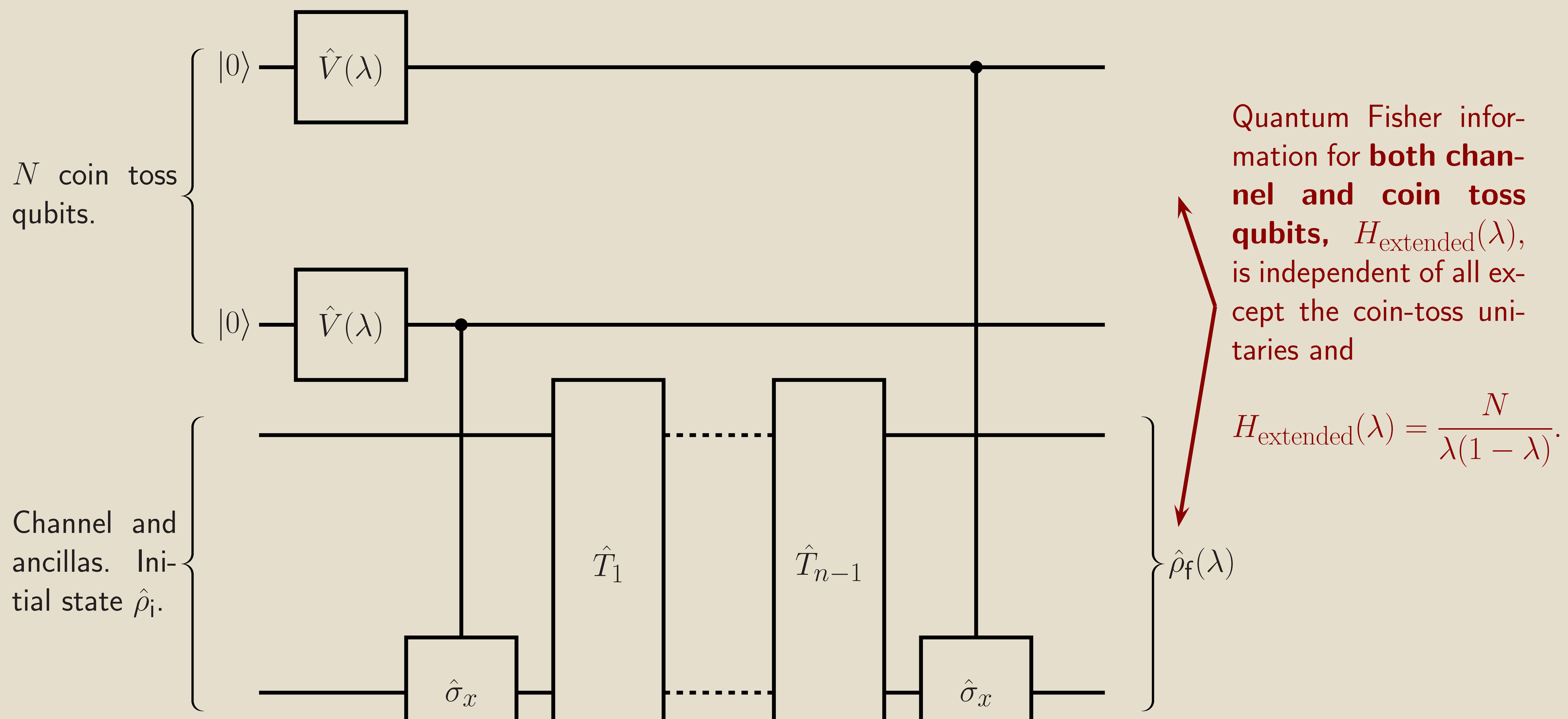
$$\hat{V}(\lambda) := \begin{pmatrix} \sqrt{\lambda} & -\sqrt{1-\lambda} \\ \sqrt{1-\lambda} & \sqrt{\lambda} \end{pmatrix}.$$

Task: Determine largest quantum Fisher information, $H(\lambda)$, with N uses of the bit flip channel.

For N independent uses of the channel, each with the pure input state $|\psi_{\text{i}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$,

$$H(\lambda) = \frac{N}{\lambda(1-\lambda)}.$$

Multiple (N) uses of the bit-flip channel interspersed with arbitrary channel unitary operations ($\{\hat{T}_j\}$), each of which is independent of λ.



The channel quantum Fisher information, $H(\lambda)$, for measurements restricted to just the channel qubits satisfies

$$H(\lambda) \leqslant H_{\text{extended}}(\lambda) = \frac{N}{\lambda(1-\lambda)}.$$

The optimal quantum Fisher information can be attained with pure channel input states with no entanglement between channel qubits.

Unitary Parameter Estimation

For any **unitary transformation** of the form $\hat{U} = e^{-i\hat{G}\lambda/2}$, where $\hat{G}$ is an Hermitian generator, the maximal quantum Fisher information using N unitary invocations is [3]:

Unentangled input states	$\Rightarrow$	$H(\lambda) = N\left(g_{\text{max}} - g_{\text{min}}\right)^2$
Entangled input states	$\Rightarrow$	$H(\lambda) = N^2\left(g_{\text{max}} - g_{\text{min}}\right)^2$

where g_{max} and g_{min} are extreme eigenvalues of $\hat{G}$.

Entanglement offers an improvement, quadratic in N, in the accuracy of unitary parameter estimation.

Parameter Estimation for Other Non-Unitary Channels

For programmable channels (e.g. bit-flip, phase-flip, and depolarizing channels) [2]:

- the optimal quantum Fisher information with N channel uses scales as $O(N)$, but
- for the depolarizing channel entanglement offers advantages [4, 5, 6].

Entanglement offers advantages for depolarizing channel parameter estimation but these do not scale beyond $O(N)$.

Conclusions

The accuracy of channel parameter estimation can be assessed via the optimal quantum Fisher information, $H(\lambda)$. For N uses of:

- the bit-flip channel, $H(\lambda) = N/\lambda(1-\lambda)$ and is attained with unentangled states.
- a unitary channel, $H(\lambda)$ scales as $O(N^2)$ and is attained with entangled states.
- the depolarizing channel, $H(\lambda)$ scales as $O(N)$ and entangled states offer advantages.

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