Enhanced Noisy Depolarizing Channel Parameter Estimation

David Collins and Jaimie Stephens

Department of Physical and Environmental Sciences, Colorado Mesa University, Grand Junction, CO 81501, USA

Abstract

We consider protocols for estimating the parameter which characterizes a single qubit depolarizing channel, with the goal of attaining the most accurate estimate per channel use. The accuracy of any quantum estimation protocol will be quantified via the quantum Fisher information (QFI) since the Crámer-Rao bound implies that a larger QFI yields a smaller lower bound on the possible variance in any estimate of the parameter. Within this framework, the choice of input state prior to channel invocation affects estimation accuracy. The known optimal estimation scheme uses pairs of maximally entangled pure input states, giving a gain over any protocol using unentangled input states. Entangling more than two qubits initially in a pure state provides no further gain in the QFI per channel use.

We ask if, *when the available input states are not pure,* these gains persist and if correlating more than two qubits is advantageous. We present a protocol using input states correlated over any number of qubits and compare this to an independent channel use protocol using uncorrelated mixed states. We show that the correlated state protocol yields gains in the QFI per channel use for certain physically reasonable parameter ranges. We show that, unlike the pure state case, using more than two correlated qubits can be advantageous and we show that, as the initial qubit states become highly mixed, adding additional correlated qubits can provide substantial gains in estimation accuracy. We show that for two qubits, such gains are attained even when the state prior to channel invocation are separable.

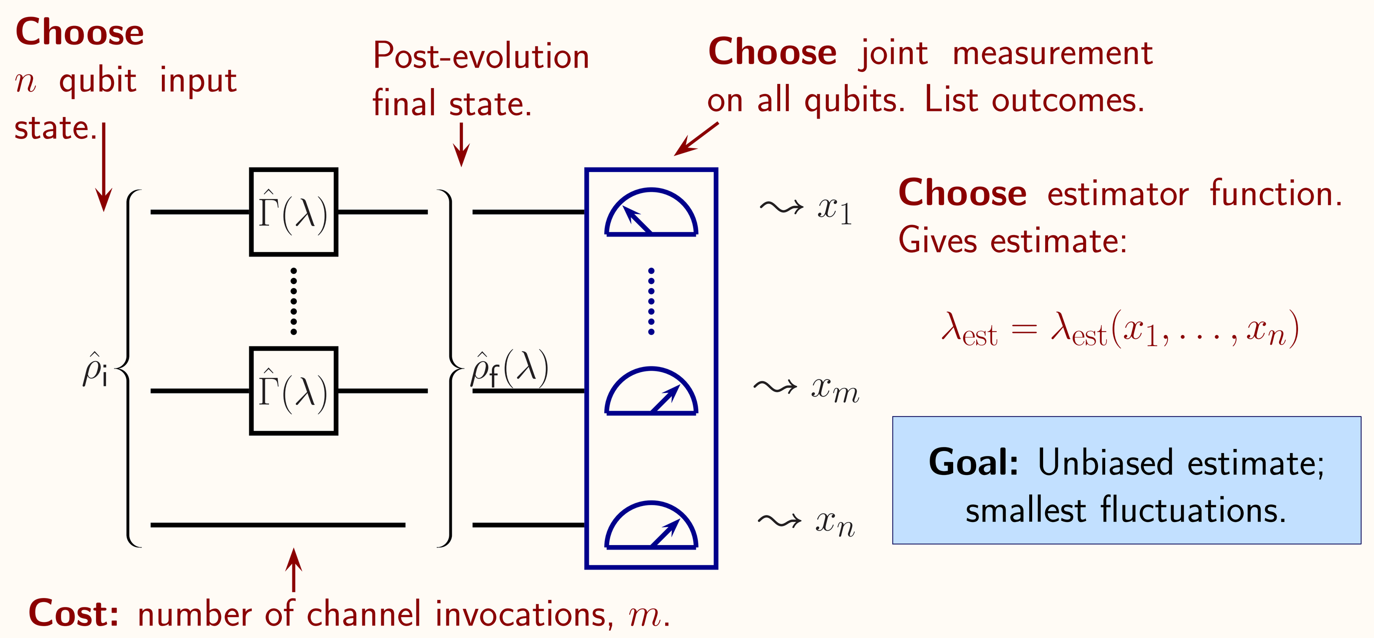
The **depolarizing channel** maps a qubit as

$$\hat{\rho} \overset{\hat{\Gamma}(\lambda)}{\mapsto} \frac{1-\lambda}{2}\,\mathrm{Tr}\,(\hat{\rho})\hat{I} + \lambda\hat{\rho}$$

where the parameter $0 \leqslant \lambda \leqslant 1$ describes the strength of the channel.

Task: Estimate parameter, λ, by subjecting quantum systems to channel.

Quantum Estimation and Fisher Information



The accuracy of the measurement is quantified in terms of the mean square error,

$$\mathrm{m.s.e}\,(\lambda_\mathrm{est}) = \left\langle \left(\lambda_\mathrm{est} - \lambda\right)^2 \right\rangle.$$

For any unbiased estimator, the **Cramér-Rao bound** gives [1]:

MSE depends on estimator choice.

$$\mathrm{m.s.e}\,(\lambda_\mathrm{est}) \geqslant \frac{1}{F(\lambda)} \geqslant \frac{1}{H(\lambda)}.$$

Quantum Fisher info. depends on $\hat{\rho}_\mathrm{f}$ but not measurement choice nor estimator.

Classical Fisher information depends on measurement choice but not estimator.

The **quantum Fisher information (QFI)** is given by

$$H(\lambda) = \mathrm{Tr}\left(\hat{\rho}_\mathrm{f}\hat{L}^2\right) \qquad \text{with} \qquad \frac{\partial\hat{\rho}_\mathrm{f}}{\partial\lambda} = \frac{1}{2}\left(\hat{\rho}_\mathrm{f}\hat{L} + \hat{L}\hat{\rho}_\mathrm{f}\right)$$

and computed [1] via diagonal decomposition, $\hat{\rho}_\mathrm{f} = \sum_k p_k \left|\phi_k\right\rangle\left\langle\phi_k\right|$, and

$$H(\lambda) = \sum_k \left(\frac{1}{p_k}\frac{\partial p_k}{\partial\lambda}\right)^2 p_k + 2\sum_{j,k}\frac{(p_j - p_k)^2}{p_j + p_k}\left|\left\langle\phi_k\right|\frac{\partial}{\partial\lambda}\left|\phi_j\right\rangle\right|^2.$$

Task: For fixed number of channel invocations, m, choose number of qubits, n, and their input state, $\hat{\rho}_\mathrm{i}$, to maximize QFI.

If the channel is invoked m times on identically prepared independent or uncorrelated quantum systems, then $H(\lambda) = mH_\mathrm{s}(\lambda)$ where $H_\mathrm{s}(\lambda)$ is the quantum Fisher information a single system/invocation. However, using **entangled or correlated input states** can, with the same number of channel invocations, **yield a larger QFI** for certain scenarios [2, 3, 4, 5].

For the depolarizing channel the **optimal protocol**, per channel invocation, uses a single channel invocation on one of a pair maximally entangled **pure input states** [3]; this **requires pure input states** and gives a slight advantage but **entangling more than two pure state qubits does not help.**

Noisy depolarization channel parameter estimation: What if pure input states are unavailable and input states must be constructed from mixed initial states?

Suppose that input states must be **constructed from mixed initial states** via unitary transformations only. Each qubit is in an initial state,

$$\hat{\rho}_0 = \frac{1}{2}\left(\hat{I} + r\hat{\sigma}_n\right)$$

where r is the **polarization (purity)** and $\hat{\sigma}_n$ a Pauli operator. This is true for typical NMR ($r \approx 10^{-5}$).

Independent Channel Use Protocol

Input state is generated using only single qubit unitary operations.

Assume that the input state is $\hat{\rho}_\mathrm{i} = \hat{\rho}_0 \otimes \hat{\rho}_0 \otimes \cdots \otimes \hat{\rho}_0$. Then for m **independent channel uses,** the **optimal QFI is**

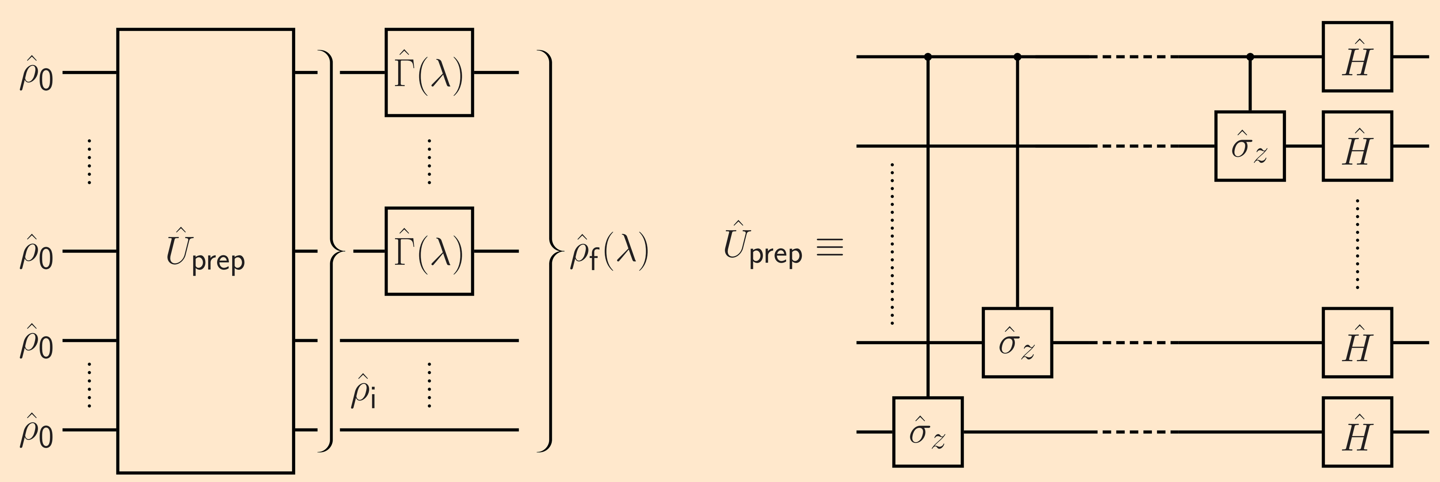
$$H_\text{opt ind}(\lambda) = m\,\frac{r^2}{1-\lambda^2 r^2}.$$

This is similar to the classical strategy of repetition and averaging. This does not give any gain in estimation accuracy, per channel use.

Could correlated input states enhance estimation accuracy (per channel use) when the qubits are initially in (noisy) mixed states? Is correlation amongst more than two qubits advantageous?

Correlated State Protocol

Input state is generated by a correlating preparatory unitary.



Same initial states as for independent channel use protocol. Same as scheme of [4, 5].

$\hat{H}$ is a single qubit Hadamard gate.

With pure initial states this would produce GHZ-type input states for the estimation process. The input state only has non-zero entries on the diagonal and counter-diagonal, facilitating diagonal decomposition. Example ($n = m = 2$):

$$\hat{\rho}_\mathrm{i} = \frac{1}{4}\begin{pmatrix} 1+r^2 & 0 & 0 & 2ir \\ 0 & 1-r^2 & 0 & 0 \\ 0 & 0 & 1-r^2 & 0 \\ -2ir & 0 & 0 & 1+r^2 \end{pmatrix} \quad \Rightarrow \quad \hat{\rho}_\mathrm{f}(\lambda) = \frac{1}{4}\begin{pmatrix} 1+\lambda^2r^2 & 0 & 0 & 2ir\lambda^2 \\ 0 & 1-\lambda^2r^2 & 0 & 0 \\ 0 & 0 & 1-\lambda^2r^2 & 0 \\ -2ir\lambda^2 & 0 & 0 & 1+\lambda^2r^2 \end{pmatrix}$$

QFI and Estimation Accuracy Gains

Compare the two protocols when using the **same number of channel invocations** and the **same polarizations**.

Diagonal decomposition of $\hat{\rho}_\mathrm{f}$ allows for computation of QFI. The **gain in QFI** is

$$G(\lambda) := \frac{H(\lambda)}{H_\text{opt ind}(\lambda)}.$$

QFI correlated state protocol.

QFI independent channel use protocol.

With channel acting on all qubits, $(m = n)$

$$H = \sum_{j=1}^{n}\binom{n-1}{j-1}\frac{2a_j\left[\left(\frac{da_j}{d\lambda}\right)^2 + n^2\lambda^{2n-2}g_j^2\right] - 4n\lambda^{2n-1}g_j^2\,\frac{da_j}{d\lambda}}{a_j^2 - \lambda^{2n}g_j^2} \quad \text{with} \quad g_j := \left[(1+r)^j(1-r)^{n-j} - (1+r)^{n-j}(1-r)^j\right]/2^{n+1}$$

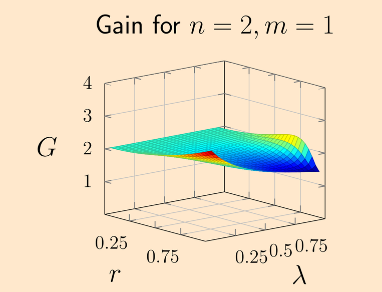
$$a_j = f_j(r) + f_j(-r) \quad \text{with} \quad f_j(r) = \frac{(1-r)^n}{2^{n+1}}\sum_{k=0}^{n}\left(\frac{1+\lambda}{2}\right)^{n-k}\left(\frac{1-\lambda}{2}\right)^k\left(\frac{1+r}{1-r}\right)^{j+k}\sum_{l=\max(0,k-n+j)}^{\min(j,k)}\binom{j}{l}\binom{n-j}{k-l}\left(\frac{1-r}{1+r}\right)^{2l}.$$

For $r \ll 1$, these give

$$H(\lambda) = n^3r^2\lambda^{2n-2} + O(r^3) \qquad \text{and} \qquad G = n^2\lambda^{2n-2} + O(r^2).$$

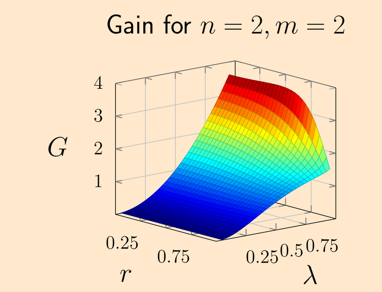
In general gain depends on number of channel invocations and polarization. **For two qubits:**

Single channel invocation



Correlated state protocol **always enhances accuracy.**

Multiple channel invocation



Correlated state protocol **enhances accuracy for certain parameters.**

Entanglement

For **two qubits** the presence of entanglement can be assessed analytically. The state of the system immediately prior to channel invocation is separable whenever $r < \sqrt{2} - 1$.

Gains in estimation accuracy cannot be attributed to entanglement.

Correlated states can enhance depolarizing channel parameter estimation accuracy:

- Enhancement depends on purity and parameter.
- Whether to use channel once or more depends on purity and parameter.
- For very small purity and weak depolarization **more than two correlated qubits can be advantageous (e.g. NMR)**, unlike pure input state case.

References

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