Correlated quantum states and enhanced mixed state Pauli channel parameter estimation

David Collins

Physical and Environmental Sciences, Colorado Mesa University, Grand Junction, CO, USA

COLORADO MESA UNIVERSITY

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Abstract

The accuracy of any physical scheme used to estimate parameters that govern the evolution of quantum systems is limited by statistical fluctuations inherent in quantum measurement processes. Quantum estimation theory provides methods for quantitative comparison of various estimation protocols. We focus on estimating the parameter governing the single qubit Pauli channel, for which it is known that optimal pure state estimation uses unentangled initial states. We consider a restricted version of this problem in which the initial states of the individual quantum systems are not pure and ask whether there are quantum estimation protocols which can yield greater accuracies than independent channel use protocols, analogous to classical repetition and averaging schemes. We compare a protocol involving quantum correlated states to independent channel use protocols. We show, that unlike the pure state case, the quantum correlated state protocol can yield greater estimation accuracy than any independent state protocol. We show that these gains persist even when the system states are separable and, in some cases, when quantum discord is absent after channel invocation.

The evolution of any quantum system is described by a quantum operation, which may depend on one or more parameters. For example, the **Pauli channel** maps a single qubit as

$$\hat{\rho} \overset{\hat{\Gamma}(\lambda)}{\mapsto} \hat{\rho}_{\mathrm{f}}(\lambda) := (1-\lambda)\,\hat{\rho} + \lambda\hat{\sigma}_n\hat{\rho}\hat{\sigma}_n$$

and depends on the parameter $0 \leqslant \lambda \leqslant 1$ and $\hat{\sigma}_n$ is a Pauli operator with n representing a direction.

Task: Knowing the direction, n, estimate parameter, λ, as accurately as possible by subjecting quantum systems to the channel.

Any physical estimation protocol requires preparation of input states for system qubits, followed by channel invocations on some or all of the qubits and terminates in measurements on qubits. The outcomes of measurements can be applied to an estimator function, which returns an estimate, λ_{est}, for λ. The probabilistic nature of outcomes of measurements on quantum systems and the effects of measurements on quantum states imply that repeated runs of any physical estimation protocol will yield estimates which fluctuate around the true parameter value.

Quantifying Estimation Accuracy: Fisher Information

The accuracy of the measurement is quantified in terms of the mean square error,

$$\mathrm{m.s.e}\left(\lambda_{\mathrm{est}}\right) = \left\langle \left(\lambda_{\mathrm{est}} - \lambda\right)^2 \right\rangle.$$

For any unbiased estimator, the **Cramér-Rao bound** gives [1]:

MSE depends on estimator choice. →

$$\mathrm{m.s.e}\left(\lambda_{\mathrm{est}}\right) \geqslant \frac{1}{F(\lambda)} \geqslant \frac{1}{H(\lambda)}.$$

← Quantum Fisher information (QFI) depends on initial state but not measurement.

↑ Classical Fisher information depends on measurement choice but not estimator.

The **quantum Fisher information** is given by

$$H(\lambda) = \mathrm{Tr}\left(\hat{\rho}_{\mathrm{f}}\hat{L}^2\right) \qquad \text{with} \qquad \frac{\partial\hat{\rho}_{\mathrm{f}}}{\partial\lambda} = \frac{1}{2}\left(\hat{\rho}_{\mathrm{f}}\hat{L} + \hat{L}\hat{\rho}_{\mathrm{f}}\right)$$

Optimal estimation: choose input state and additional parameter-independent unitaries so as to maximize the quantum Fisher information.

The quantum Fisher information can be increased by invoking the channel multiple times. If there are m invocations on identically prepared independent or uncorrelated quantum systems, then it is always true that $H(\lambda) = mH_{\mathrm{s}}(\lambda)$ where $H_{\mathrm{s}}(\lambda)$ is the quantum Fisher information for one invocation on one systems.

Quantum estimation: Given m channel invocations is it possible, by using quantum resources, to exceed the m-fold increase attained with "classical" independent uses of the channel?

The independent channel use improvement in the QFI can be enhanced by an additional factor of m for certain scenarios by using entangled or correlated states [2, 3].

Pauli Channel Parameter Estimation

In any estimation protocol using m channel invocations the QFI is bounded by [4, 5, 6]:

$$H(\lambda) \leqslant \frac{m}{\lambda(1-\lambda)}.$$

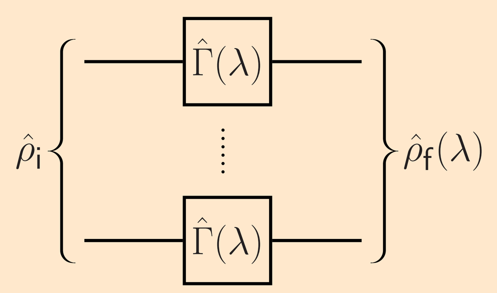
For **pure input states** there is a protocol, in which the Pauli channel is applied independently to m qubits and which saturates the bound. For **pure input states,** entanglement and quantum correlations **cannot enhance parameter estimation accuracy.**

Could correlated states enhance estimation accuracy when none of the qubits are initially in pure states?

Independent Channel Use Protocol

Assume that each qubit initially has the same polarization and that the channel is a phase flip channel.

Channel invocation on m qubits.



General product input state:

$$\hat{\rho}_{\mathrm{i}} = \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \cdots \otimes \hat{\rho}_m$$

where $\hat{\rho}_j = \frac{1}{2}\left(\hat{I} + \mathbf{r}_j \cdot \hat{\boldsymbol{\sigma}}\right)$. Polarization, $r := |\mathbf{r}_j|$, is the same for all qubits.

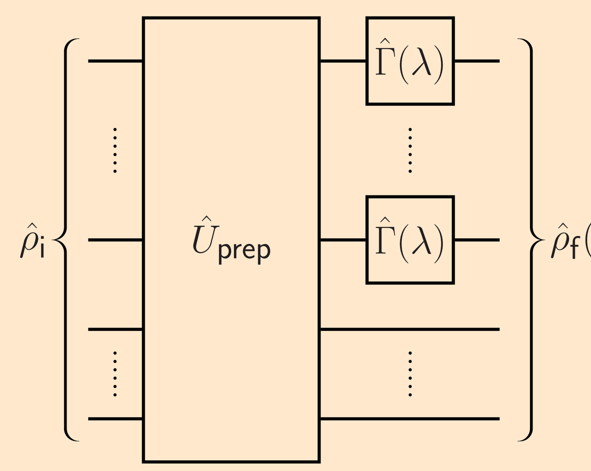
The optimal input states all have $\mathbf{r}_j = r\hat{\boldsymbol{y}}$. Then for m **independent channel uses,** the **optimal QFI is**

$$H_{\mathrm{opt\ ind}}(\lambda) = \frac{4r^2\, m}{1-(1-2\lambda)^2 r^2}.$$

The optimal independent channel use QFI **depends linearly on** m and **does not saturate the absolute upper bound on the QFI**.

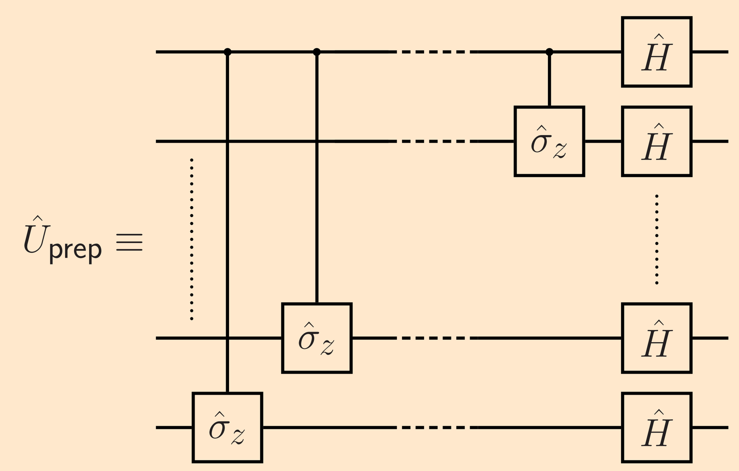
Correlated State Protocol

Channel invocation on m of n qubits is preceded by a correlating preparatory unitary.



Same product input states as for independent channel use protocol.

Preparatory unitary structure.



Here $\hat{H}$ is a single qubit Hadamard transformation.

The structure of $\hat{\rho}_{\mathrm{f}}$ allows for computation of the quantum Fisher information:

$$H(\lambda) = \frac{m^2(1-2\lambda)^{2m-2}}{2^{n-1}} \sum_{j=0}^{n} \binom{n}{j} \frac{c_{j-}^2 c_{j+}}{c_{j+}^2 - (1-2\lambda)^{2m} c_{j-}^2}.$$

where

$$c_{j\pm} = (1+r)^j(1-r)^{n-j} \pm (1+r)^{n-j}(1-r)^j.$$

Estimation Accuracy Gains

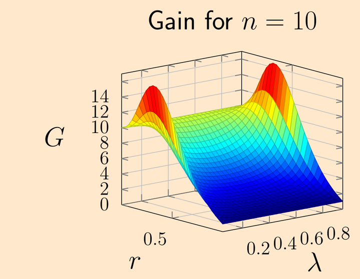
Compare the two protocols when using the **same number of channel invocations** and the **same polarizations**.

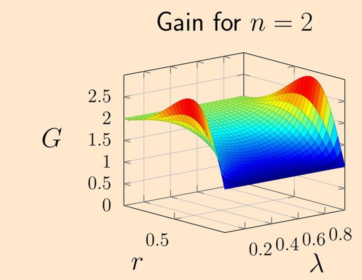
Information gain: $G(\lambda) := \dfrac{H(\lambda)}{H_{\mathrm{opt\ ind}}(\lambda)}$.

← QFI correlated state protocol.

← QFI independent channel use protocol.

Single channel invocation

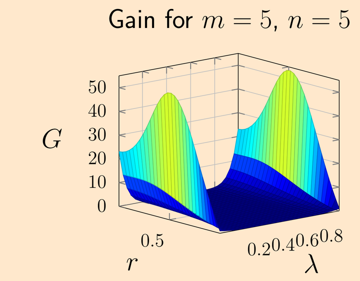


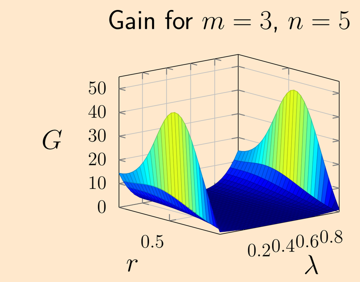


Correlated state protocol **always increases accuracy** and

$$G(\lambda) \to n \quad \text{as } r \to 0.$$

Multiple channel invocations





Correlated state protocol **does not increase accuracy throughout parameter range.** But, in some regions,

$$G(\lambda) \geqslant n.$$

Entanglement and Discord

For 2 qubits the presence of entanglement and discord can be assessed analytically.

The state of the system immediately prior to channel invocation is separable whenever $r < \sqrt{2}-1$. Whenever $r > 0$, the system has non-zero discord prior to channel invocation, but after channel invocation the discord depends on the value of λ and for $\lambda = 1/2$ the post-channel discord is zero. In contrast, for mixed state phase shift estimation using the same type of protocol discord is preserved by the channel operations and gains can be unequivocally associated with discord [3].

Gains in estimation accuracy cannot be attributed to entanglement or post-channel invocation quantum discord. They may be related to presence of discord prior to channel invocation.

Correlated states can enhance estimation accuracy for mixed state Pauli channel parameter estimation.

References

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