The Physics of Rainbows

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CMU Physics Seminar
September 13, 2012

Charles A. Bennett, *Principles of Physical Optics*, 1st ed., pps. 111-115.

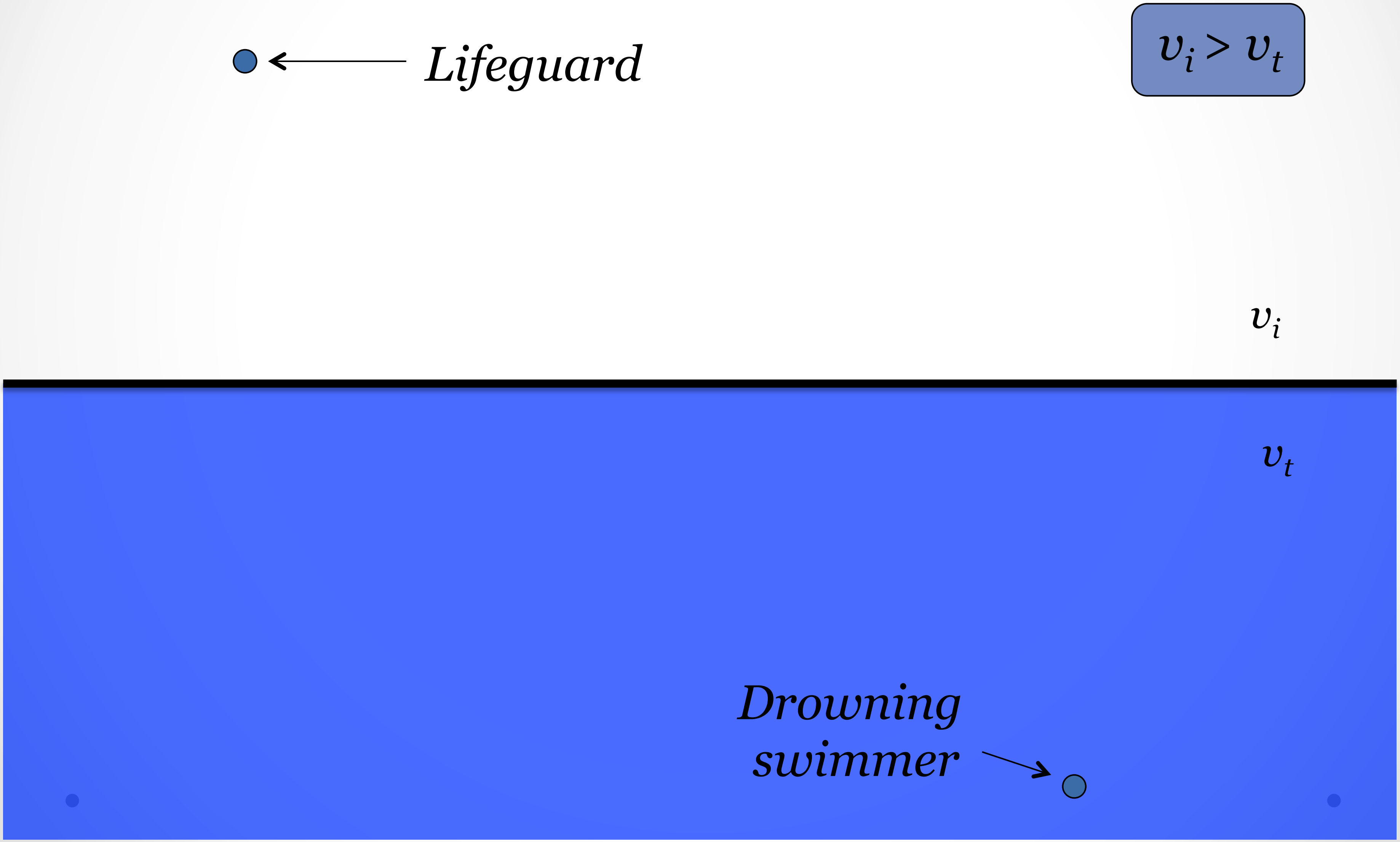
Jearl D. Walker, "Multiple rainbows from single drops of water and other liquids", *American Journal of Physics* Vol. 44, No. 5, May 1976.

John A. Adam, "The mathematical physics of rainbows and glories", *Physics Reports* 356 (2002), 229-365.

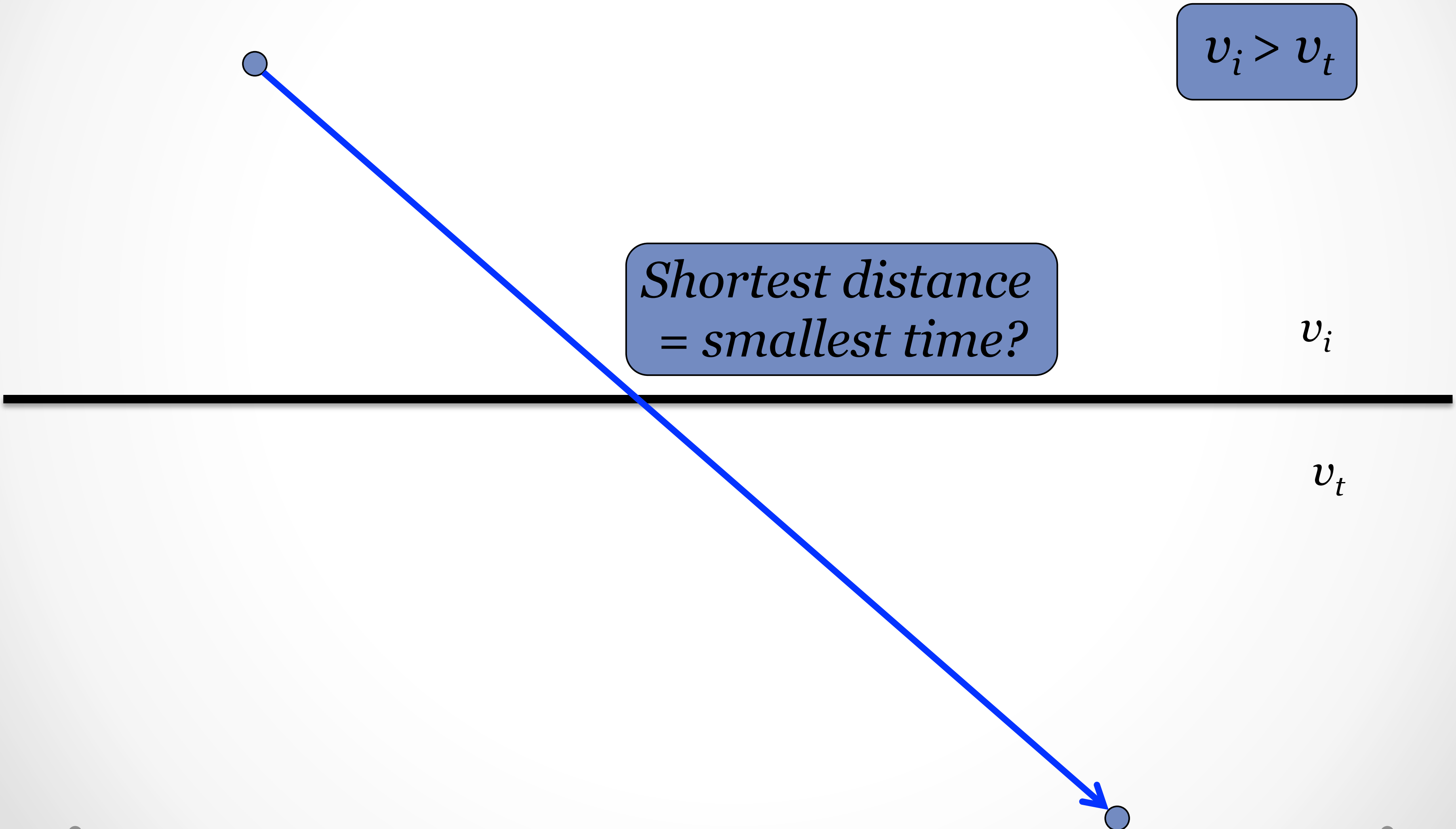
Quiz: The Physics of Rainbows

1. For the primary rainbow, what is the ‘exterior’ (*largest* observed angle) color?
   a) Violet b) Red c) Yellow d) Green
2. For the primary rainbow, what is the ‘interior’ (*smallest* observed angle) color?
   a) Violet b) Red c) Yellow d) Green
3. For the secondary rainbow, what is the ‘exterior’ color?
   a) Violet b) Red c) Yellow d) Green
4. For the secondary rainbow, what is the ‘interior’ color?
   a) Violet b) Red c) Yellow d) Green
5. Is the region between the primary and secondary rainbows *dark* or *bright*?
6. Is the interior region of the primary rainbow *dark* or *bright*?

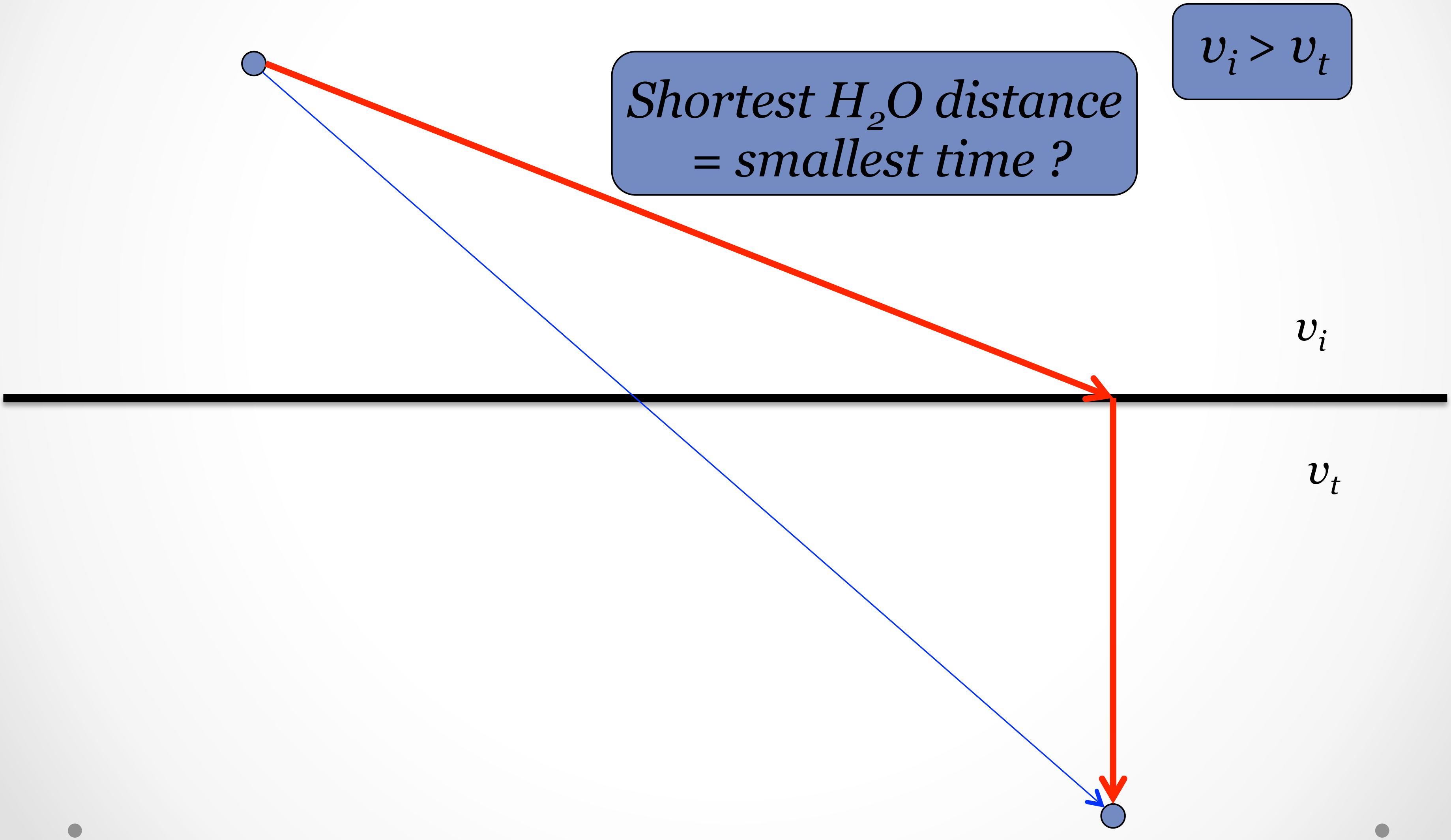
What path should the lifeguard take to *minimize* her *transit time*?



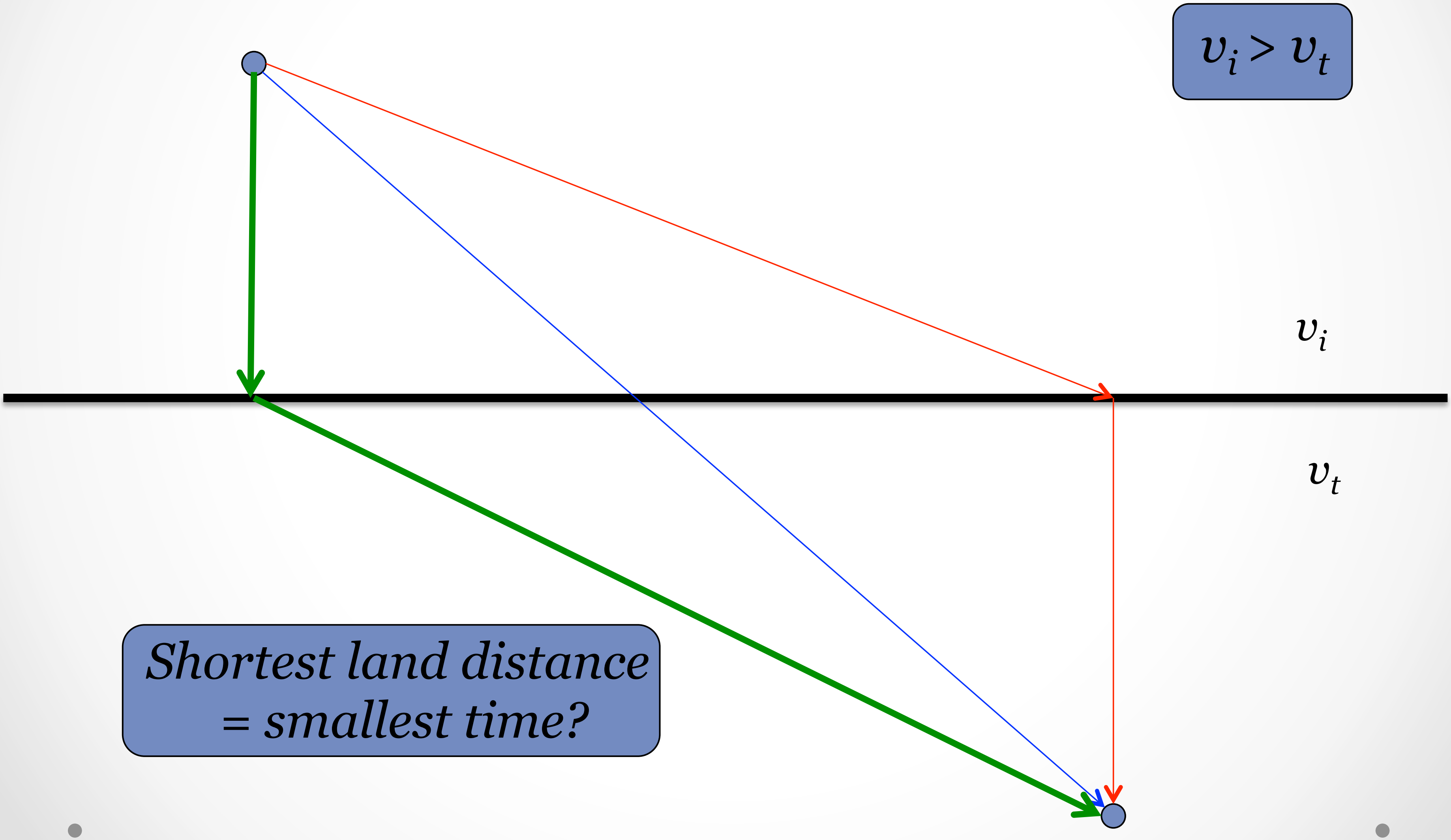
Smallest time interval?



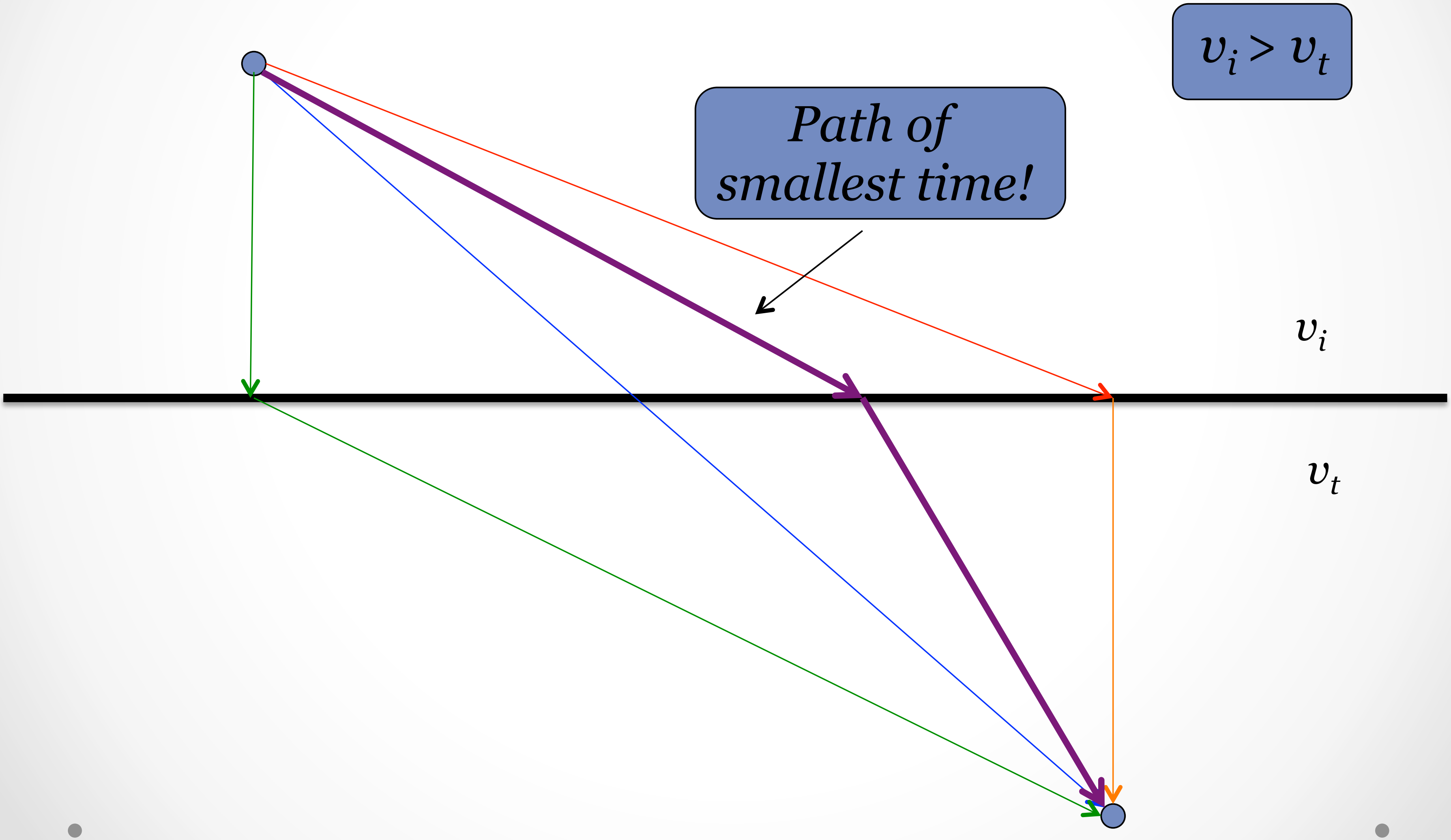
Smallest time interval?



Smallest time interval?

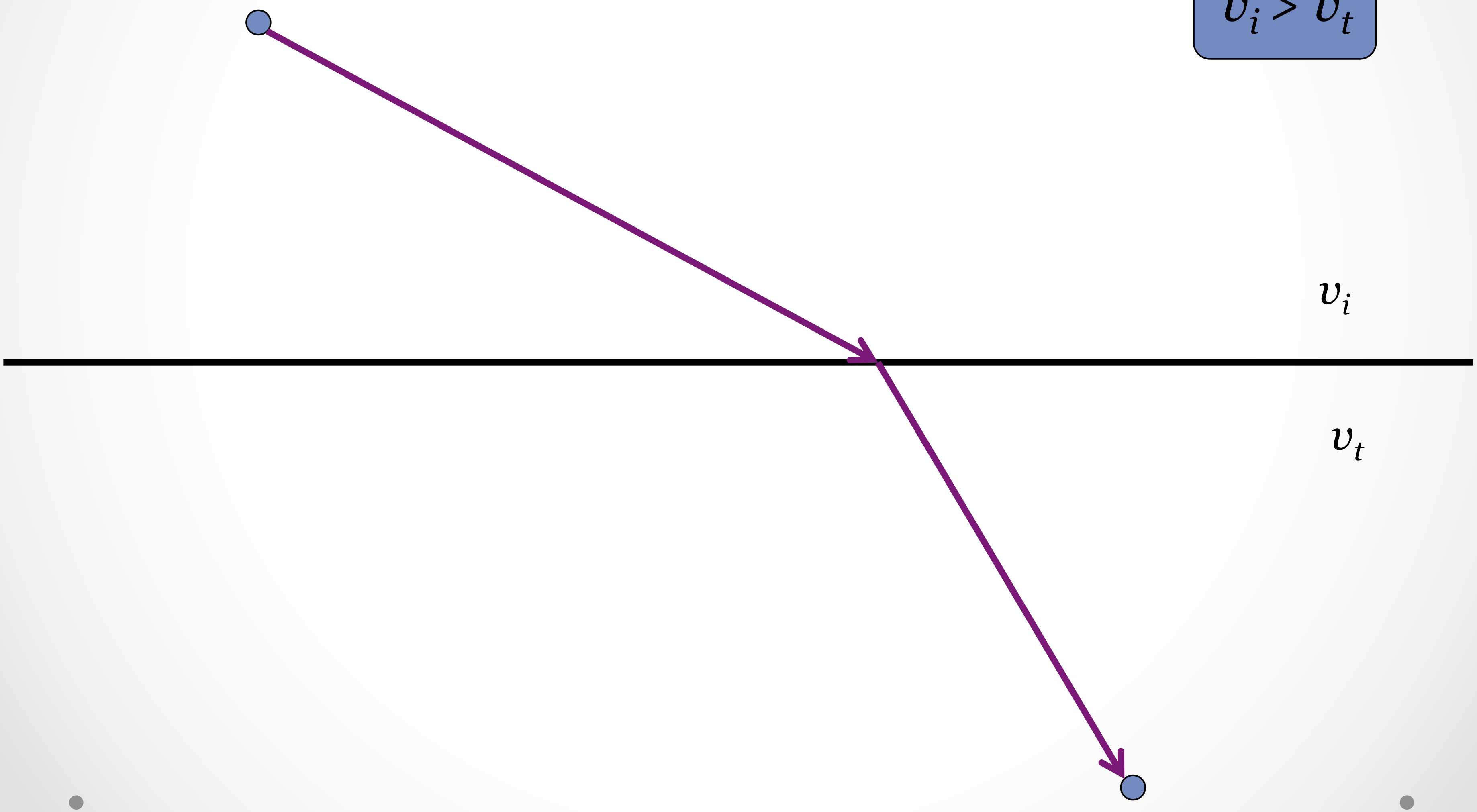


Smallest time interval?

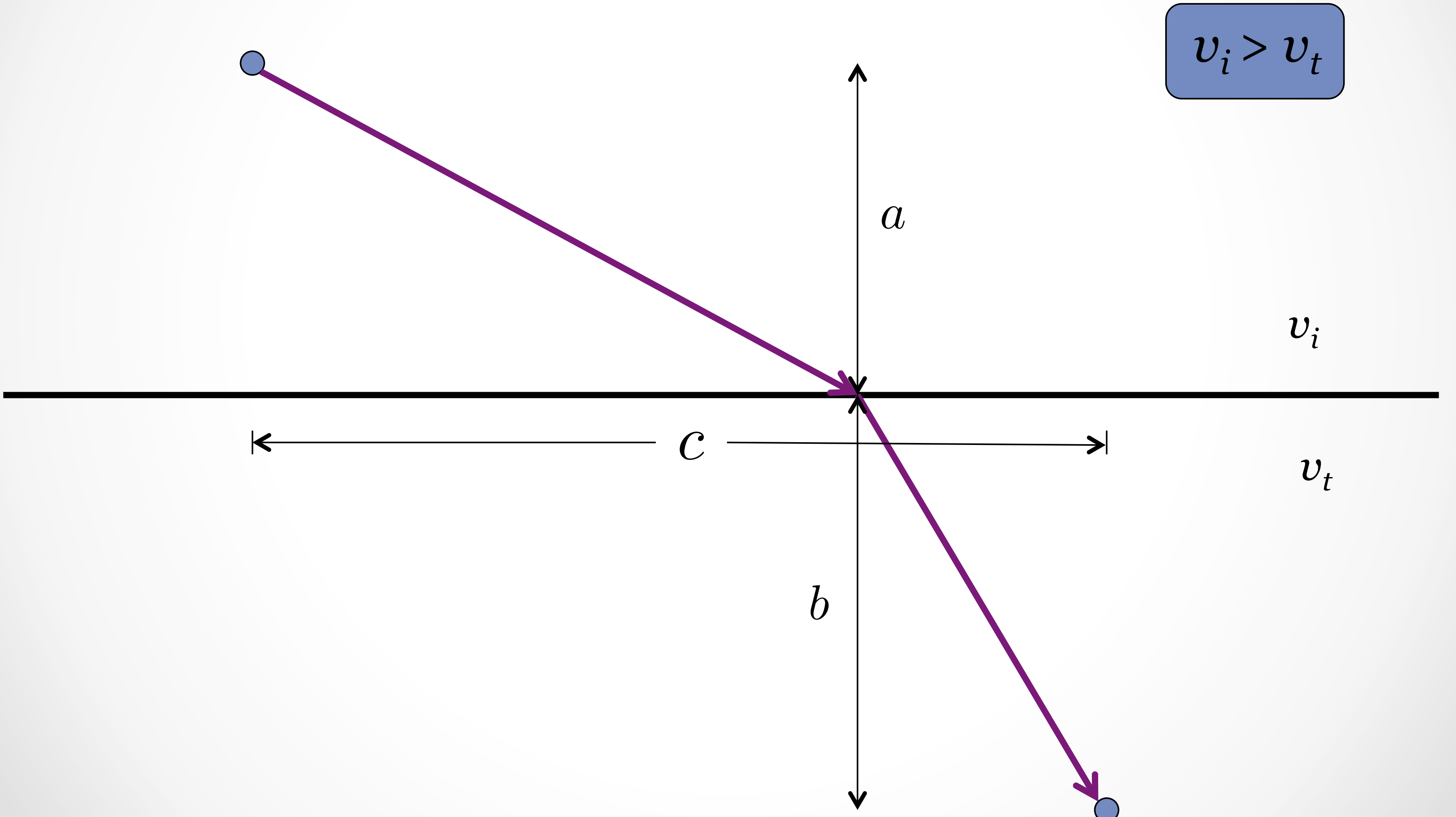


Smallest time interval?

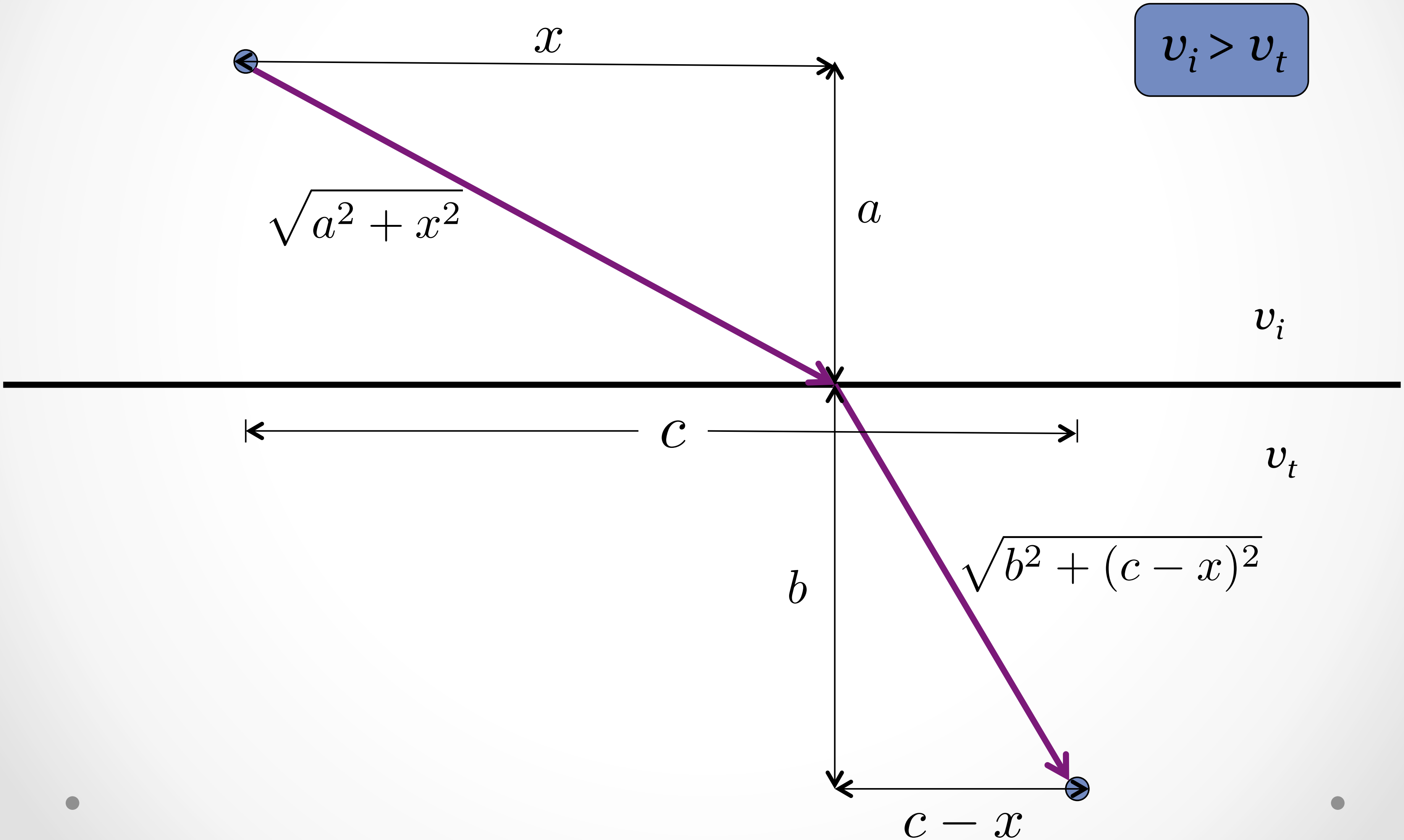
$v_i > v_t$



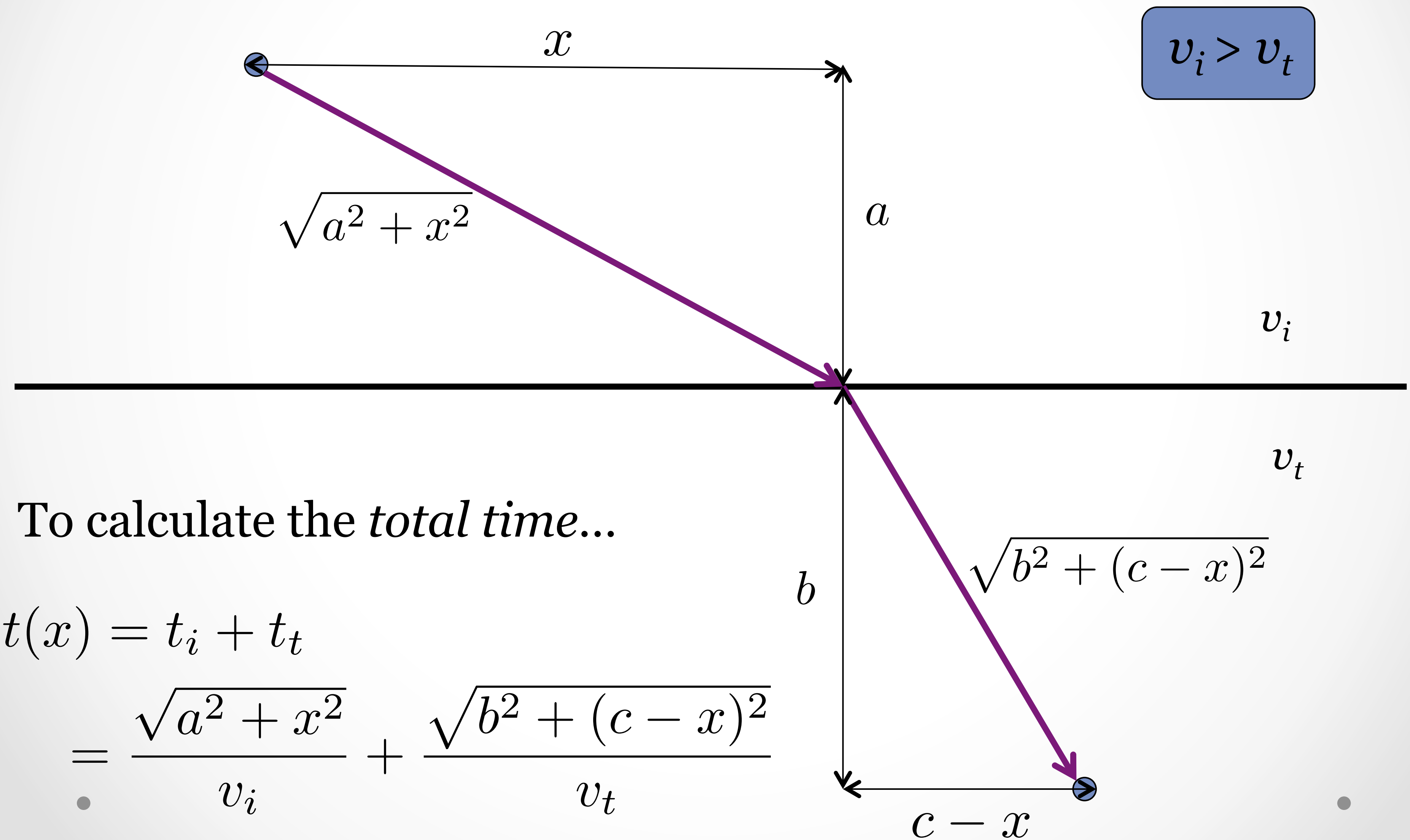
Smallest time interval?



Smallest time interval?



Smallest time interval?

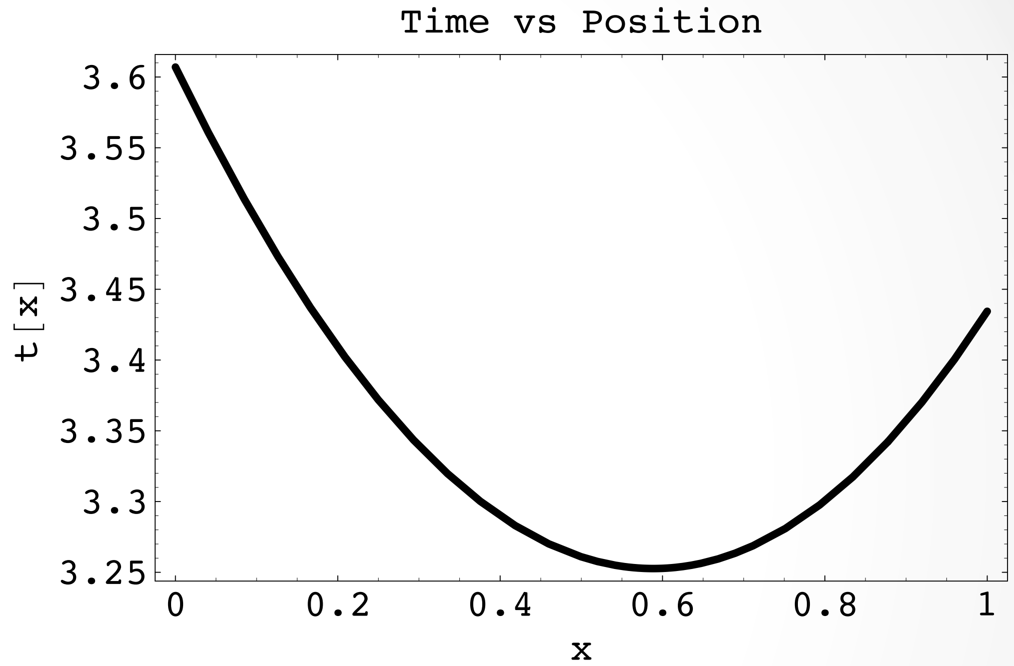


To calculate the *total time*…

$$t(x) = t_i + t_t$$

$$= \frac{\sqrt{a^2 + x^2}}{v_i} + \frac{\sqrt{b^2 + (c - x)^2}}{v_t}$$

Plot of t(x) vs x…

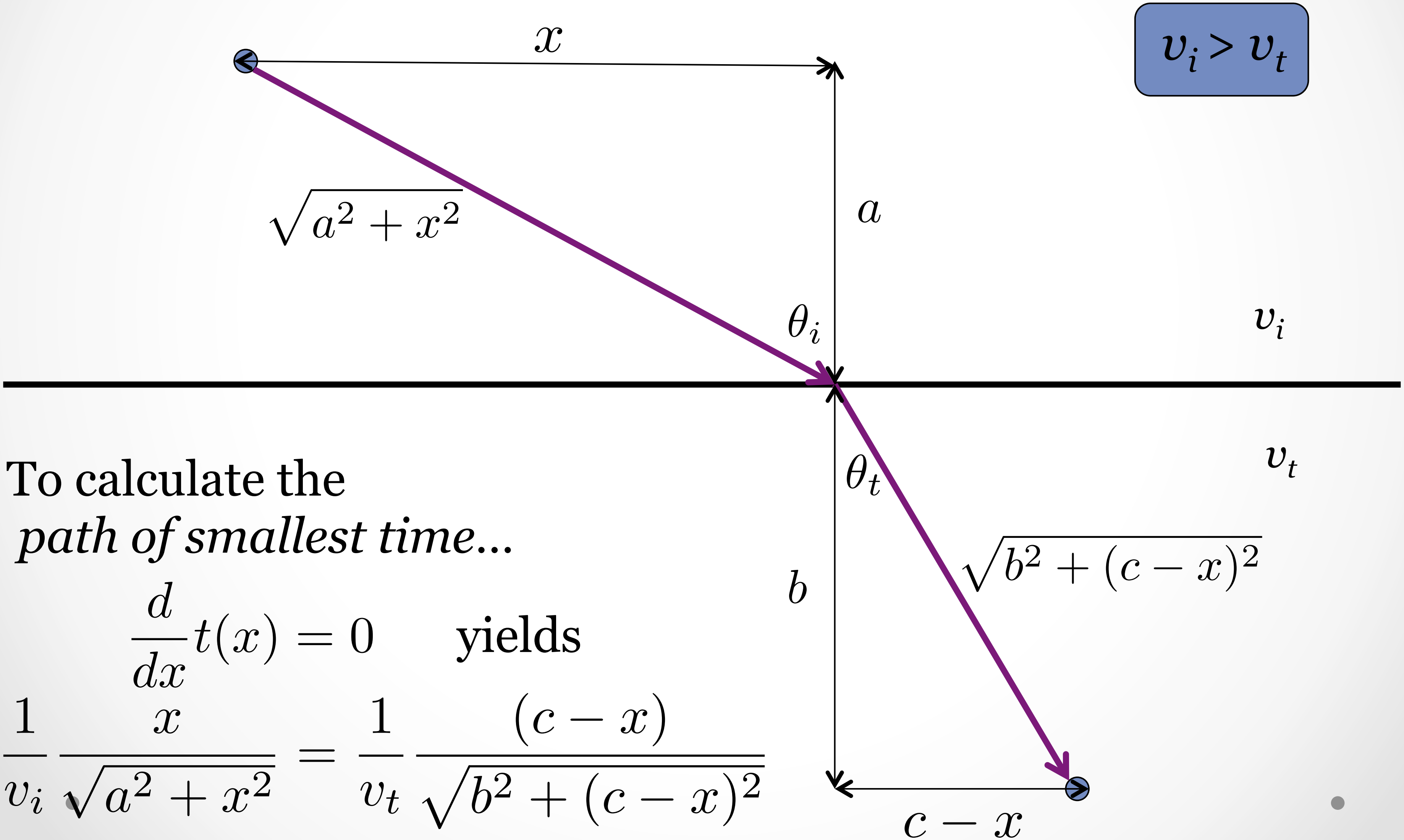


The *total time is…*

$$a, b, c = 1$$
$$v_i = 0.8$$
$$v_t = 0.6$$

$$t(x) = t_i + t_t$$
$$= \frac{\sqrt{a^2 + x^2}}{v_i} + \frac{\sqrt{b^2 + (c - x)^2}}{v_t}$$

Smallest time interval?

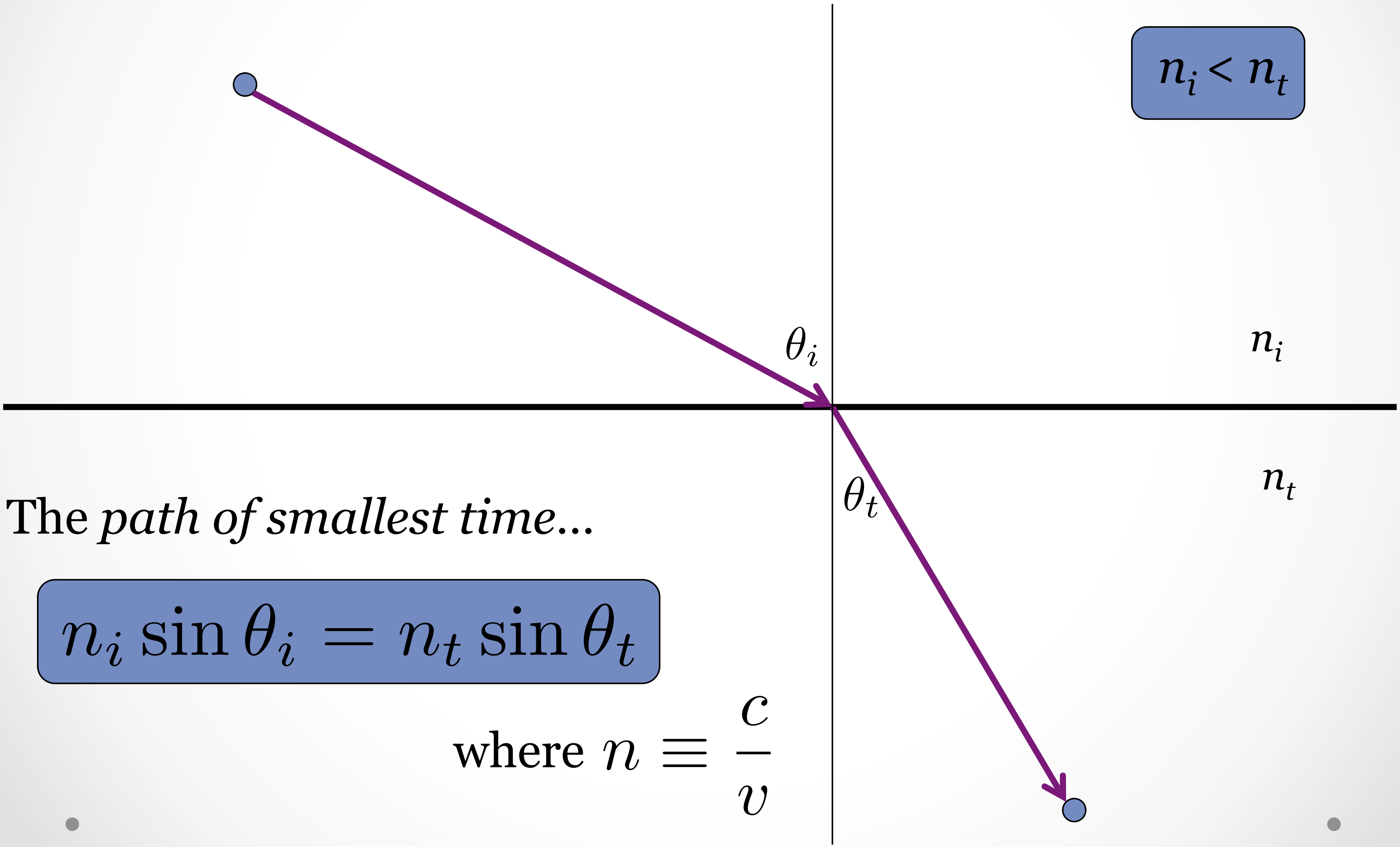


To calculate the
path of smallest time…

$$\frac{d}{dx} t(x) = 0 \qquad \text{yields}$$

$$\frac{1}{v_i} \frac{x}{\sqrt{a^2 + x^2}} = \frac{1}{v_t} \frac{(c - x)}{\sqrt{b^2 + (c - x)^2}}$$

Smallest time interval?



The *path of smallest time…*

$$n_i \sin\theta_i = n_t \sin\theta_t$$

where $n \equiv \dfrac{c}{v}$

Fermat's Principle:

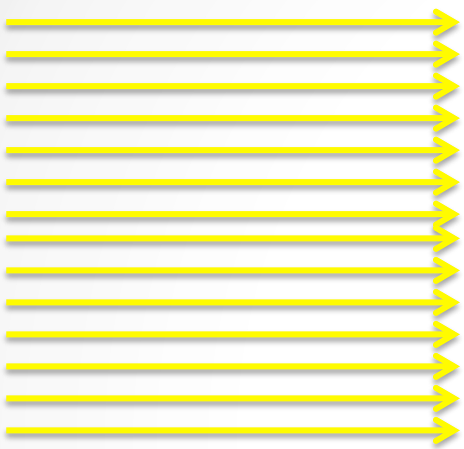
The actual path between two points taken by a beam of light is the one that is traversed in the *least time*.

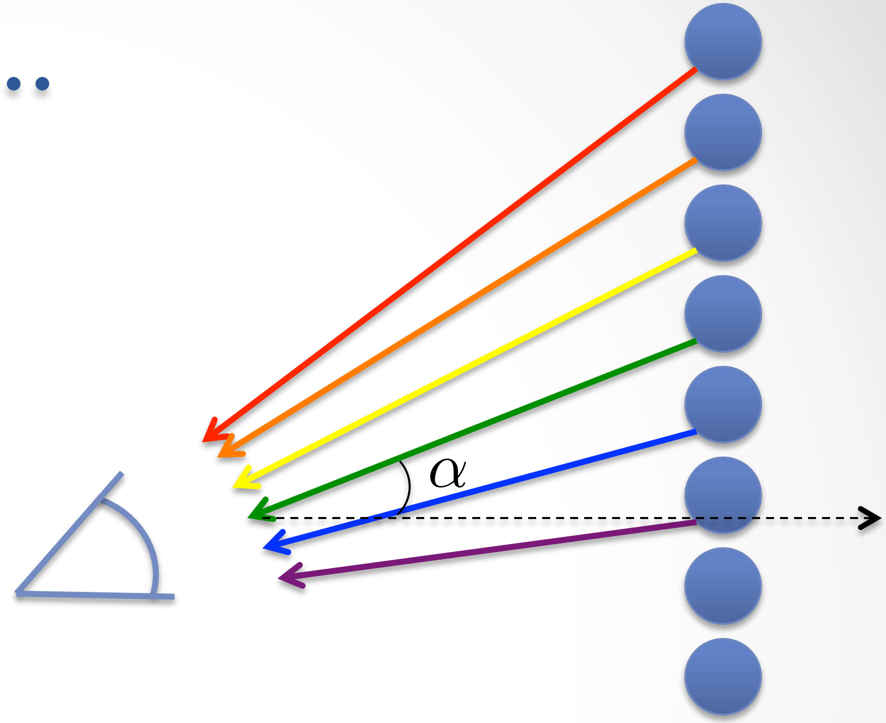
When light enters a new medium, it's path obeys Snell's Law:

$$n_i \sin\theta_i = n_t \sin\theta_t$$

Geometrical setup…

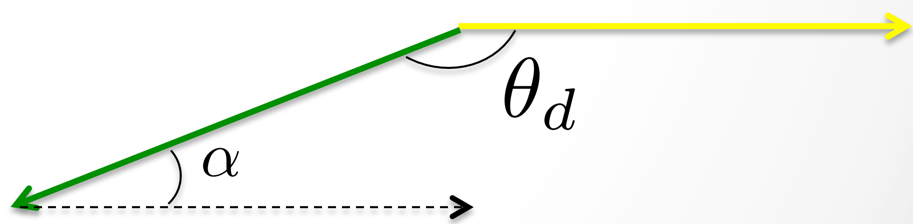
Sunlight





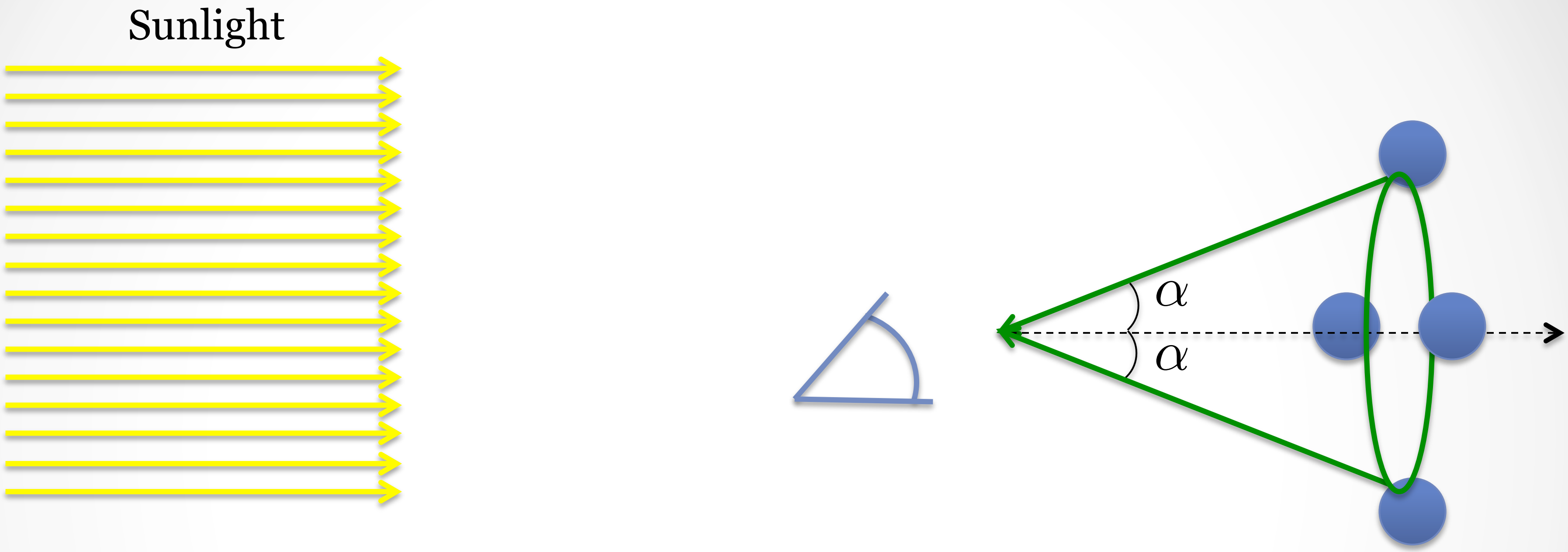
- Rays of sunlight are *nearly parallel* to each other.

- Suspended H_2O droplets are *nearly spherical* due to *surface tension.*



- θ_d is the *deviation angle.*
- α is the *observed angle*, measured from the *anti-solar* direction.

Geometrical setup…

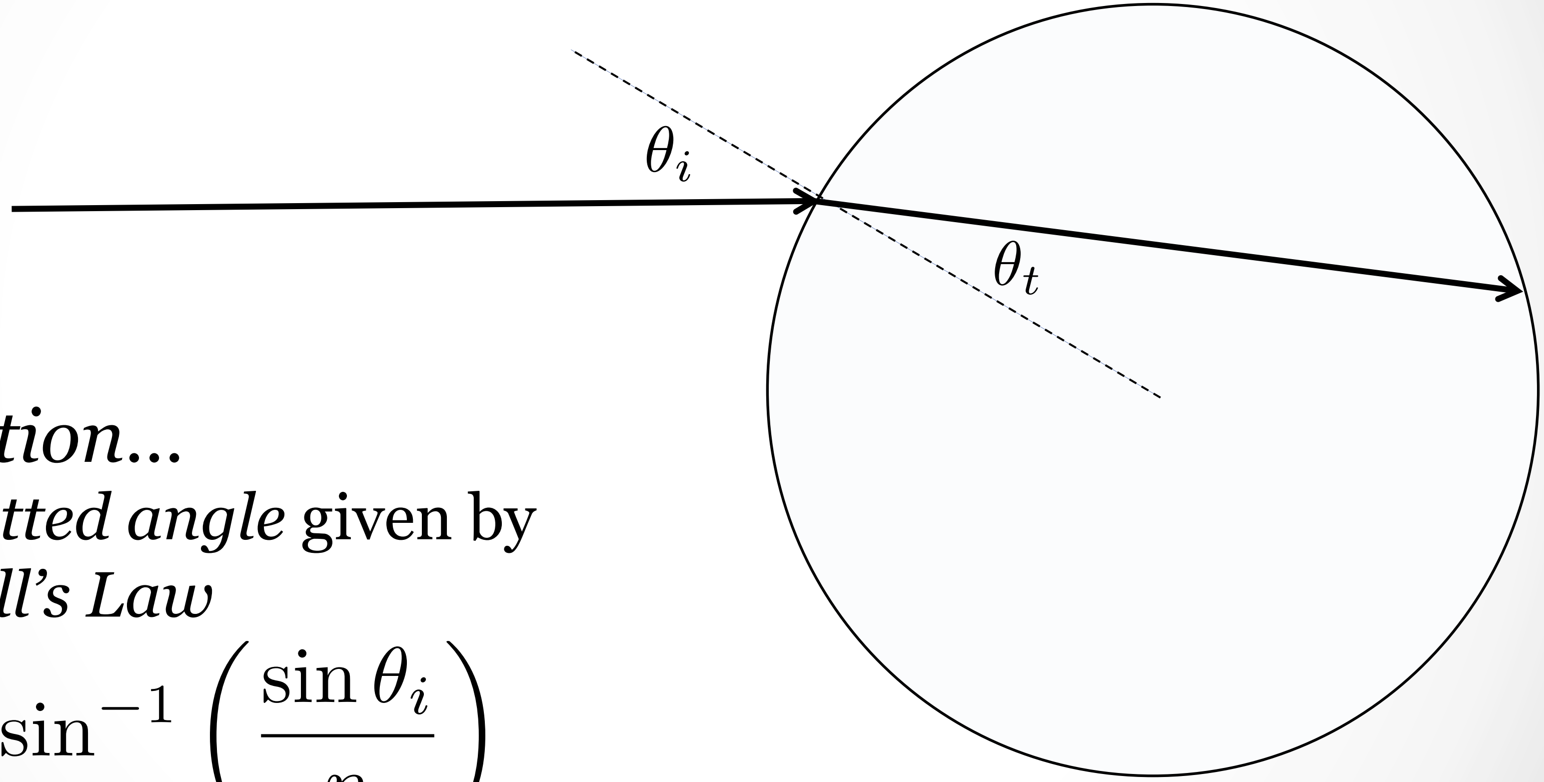


- At certain *observed angles*, a particular color will dominate.
  - This angle forms a *cone* around the *anti-solar* direction.
  - All raindrops that lie on this cone can contribute to the rainbow

=> The rainbow is *circular!*

Primary Rainbow:

Light is incident at angle θ_i ...



1st refraction...

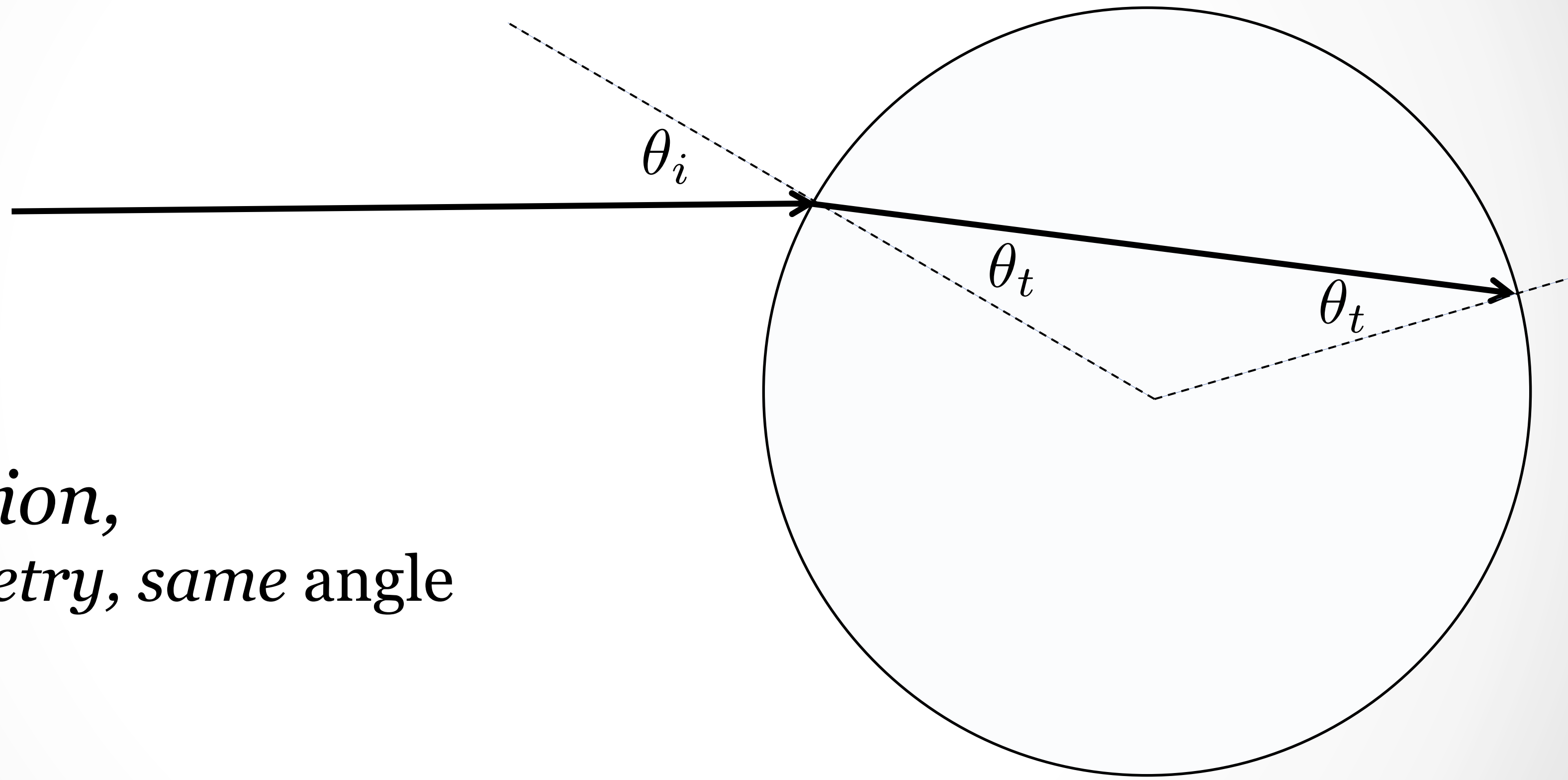
- *transmitted angle* given by *Snell's Law*

$$\theta_t = \sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$

where $n \equiv \frac{n_t}{n_i}$

$$n_{air} \simeq 1.00$$
$$n_{H2O} \simeq 1.33$$

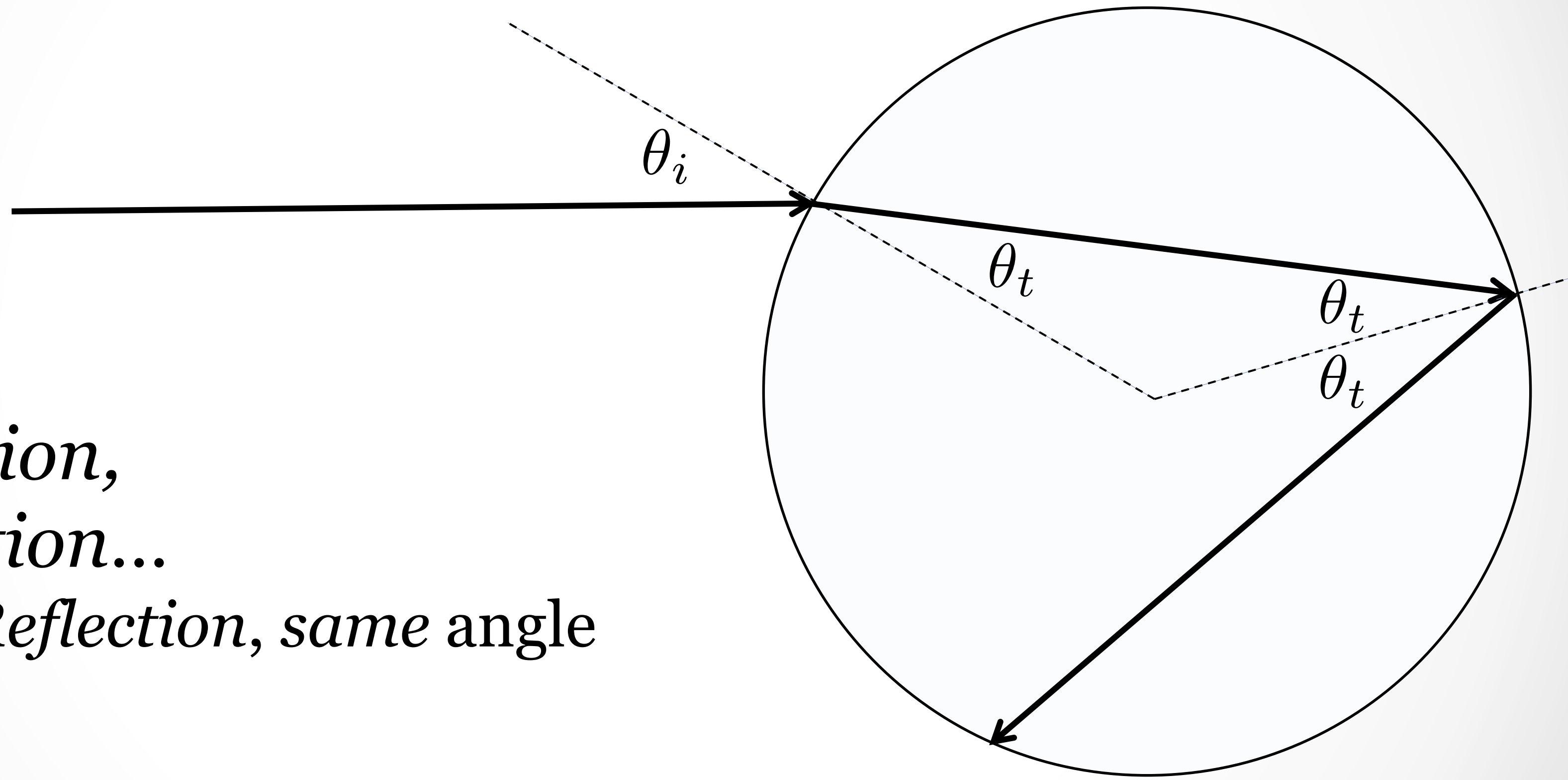
Primary Rainbow:



1st refraction,

- by *geometry, same* angle

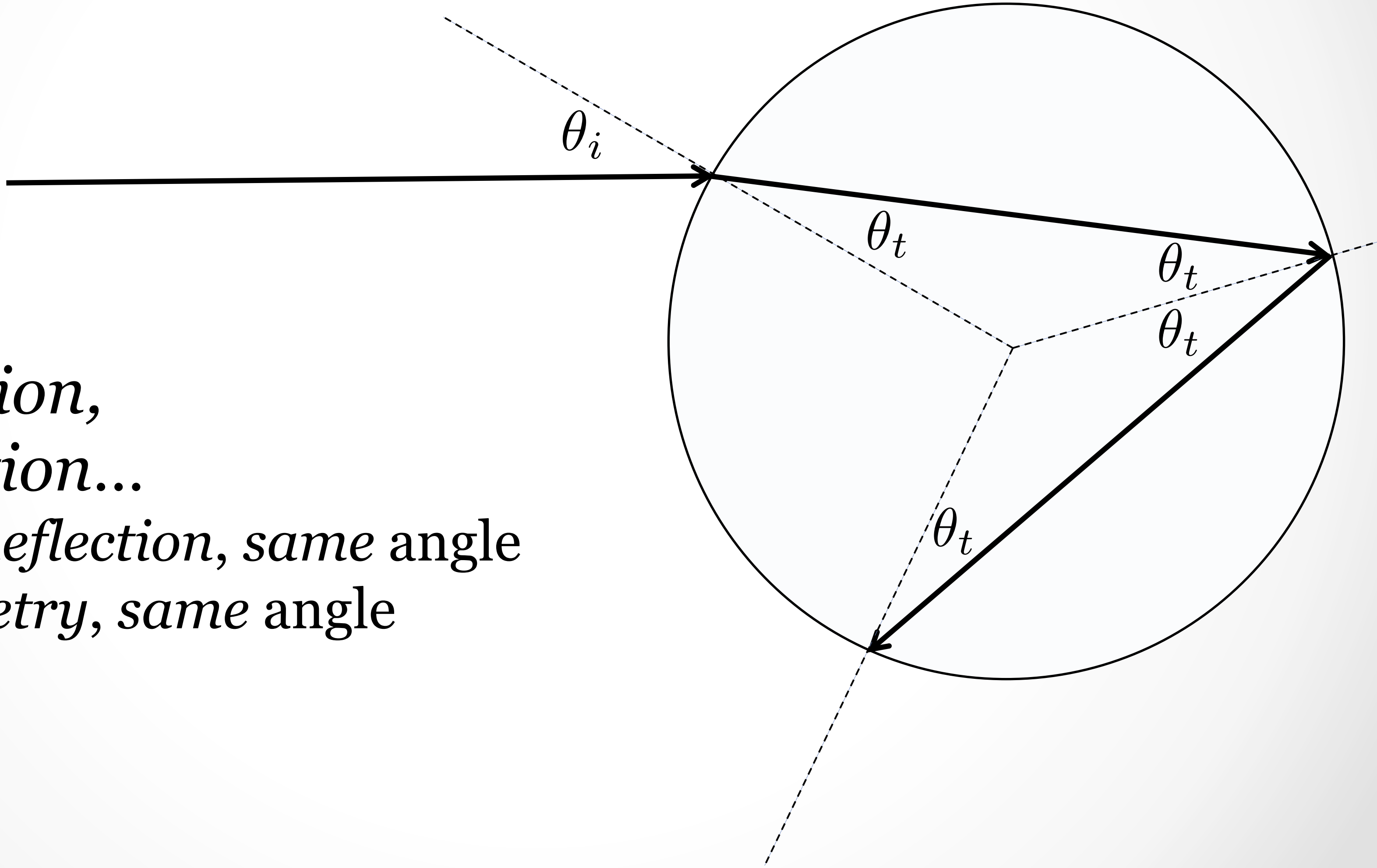
Primary Rainbow:



1st refraction,
2nd reflection…

- *Law of Reflection, same* angle

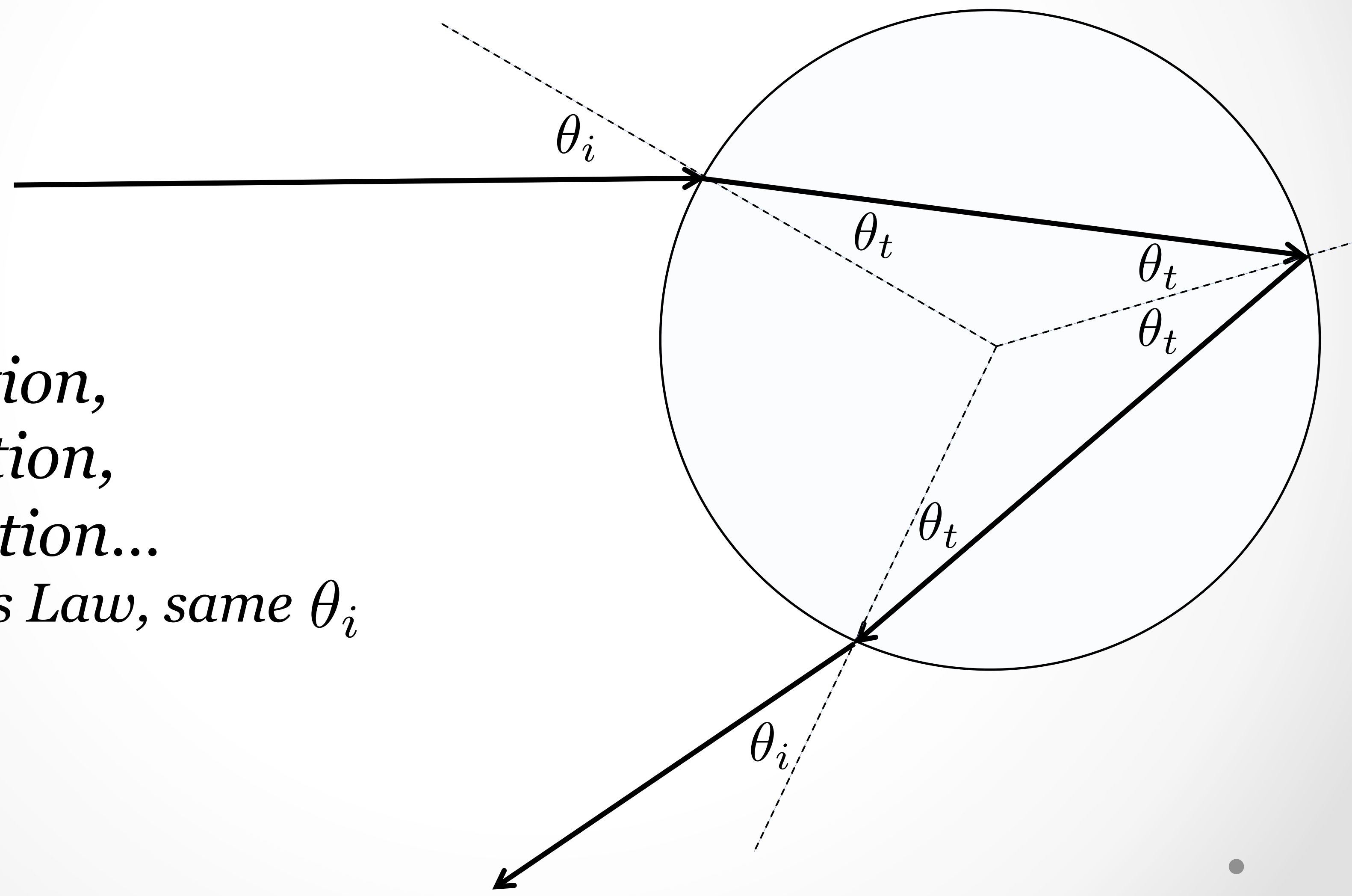
Primary Rainbow:



1st refraction,
2nd reflection…

- *Law of Reflection, same* angle
- by *geometry, same* angle

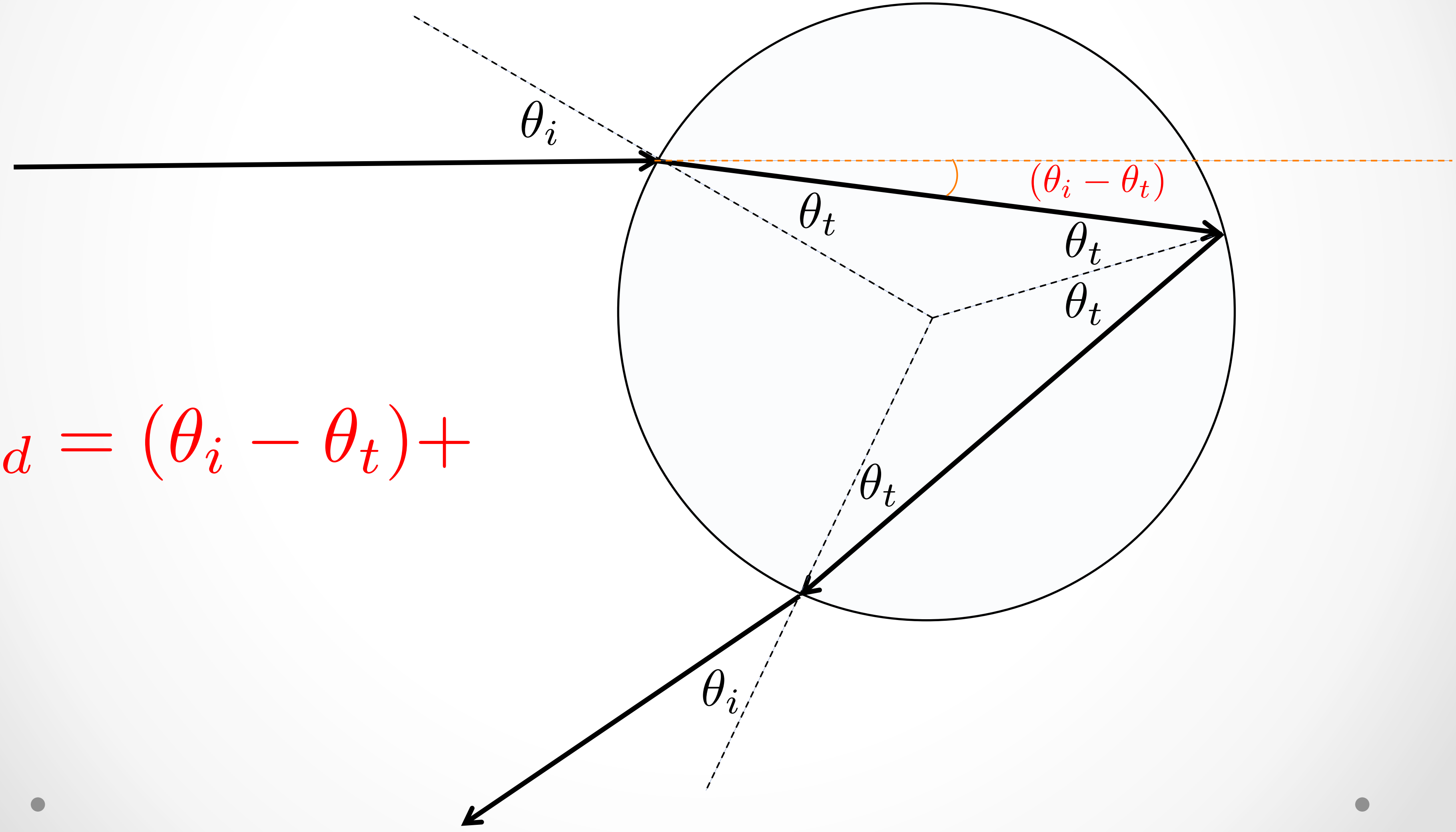
Primary Rainbow:



1st refraction,
2nd reflection,
3rd refraction…

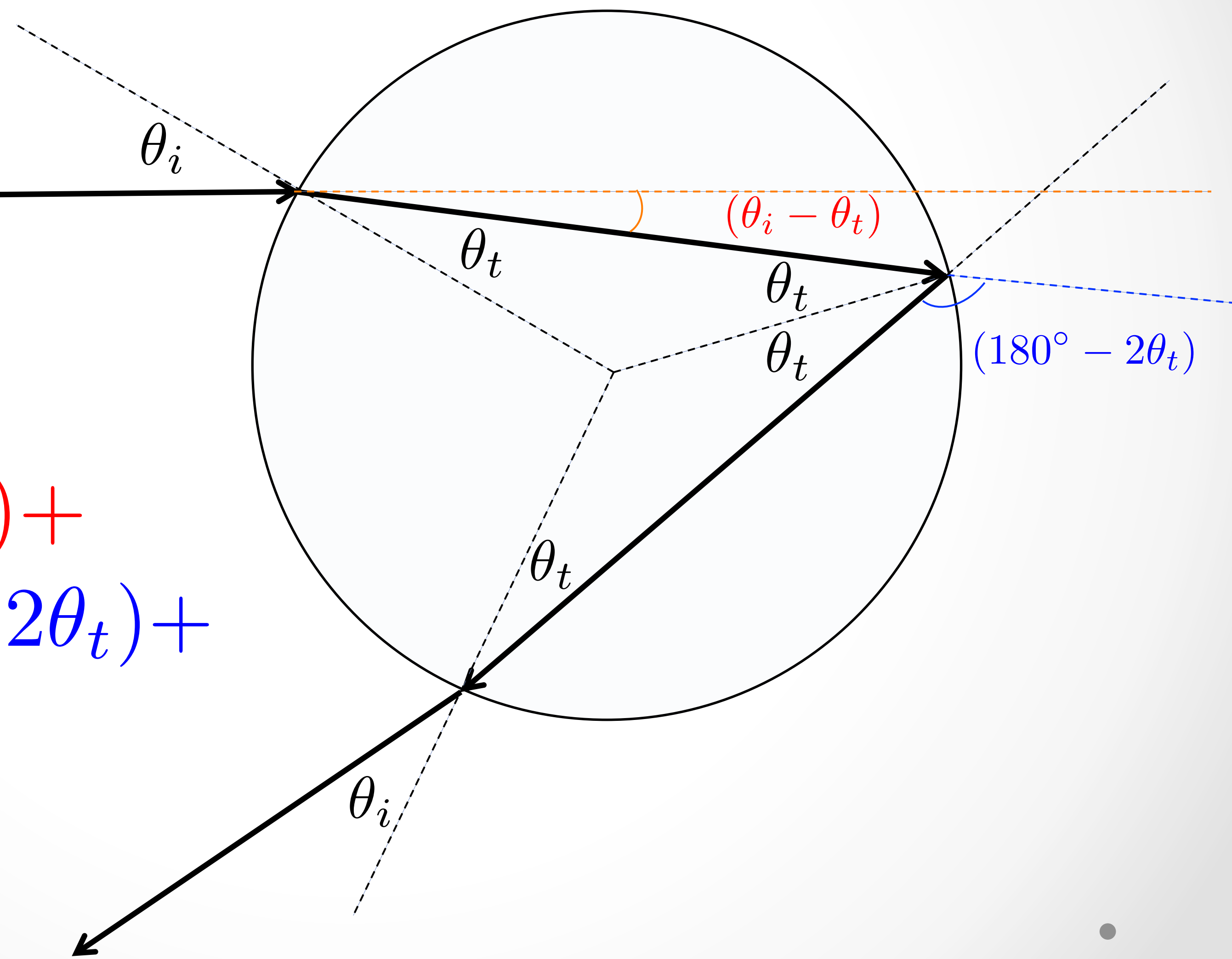
- by *Snell's Law, same* θ_i

Calculate the *net deviation angle*, θ_d ...



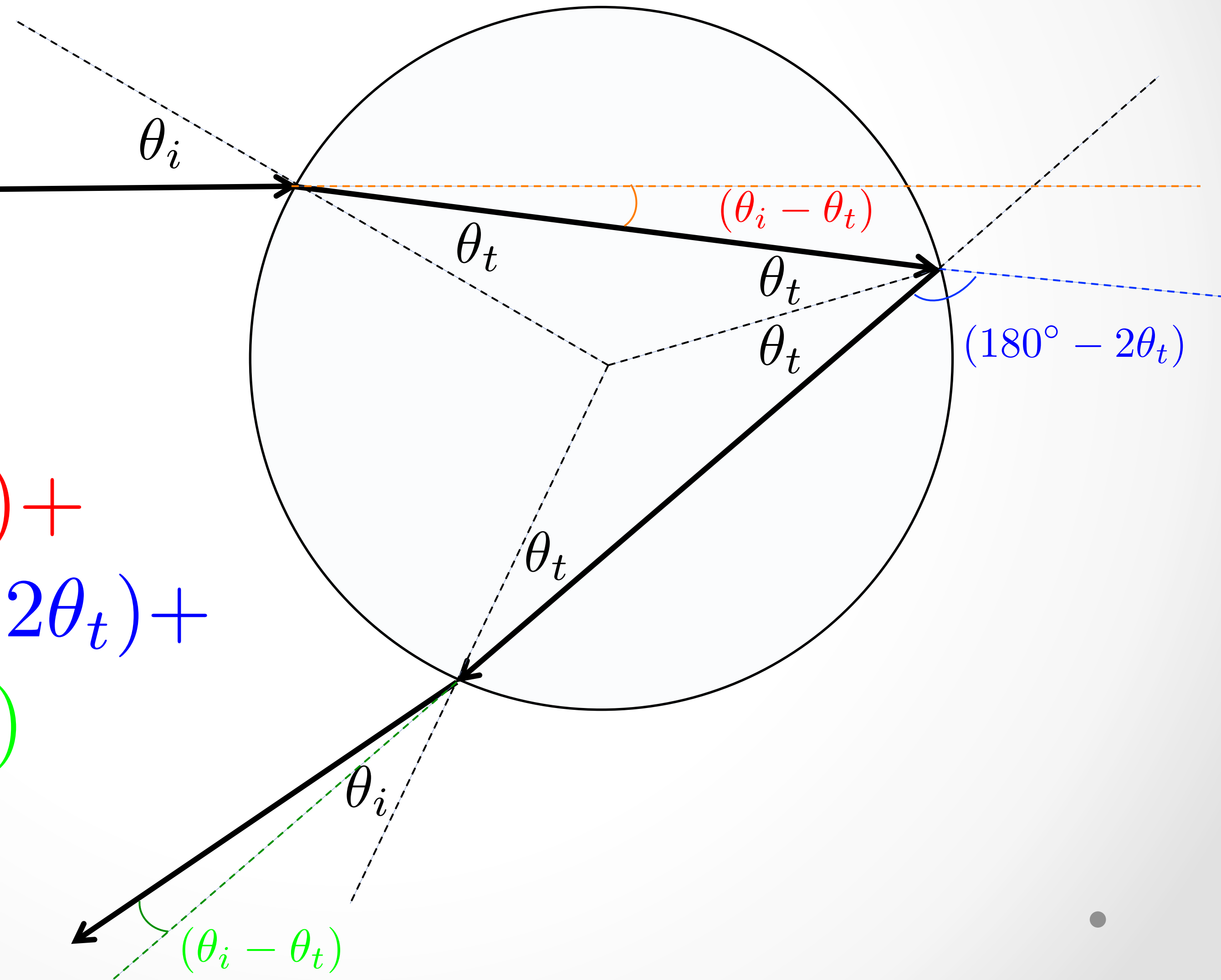
$$\theta_d = (\theta_i - \theta_t) +$$

Calculate the *net deviation angle*, θ_d ...



$$\theta_d = (\theta_i - \theta_t) + (180° - 2\theta_t) +$$

Calculate the *net deviation angle*, θ_d ...



$$\theta_d = (\theta_i - \theta_t) + (180° - 2\theta_t) + (\theta_i - \theta_t)$$

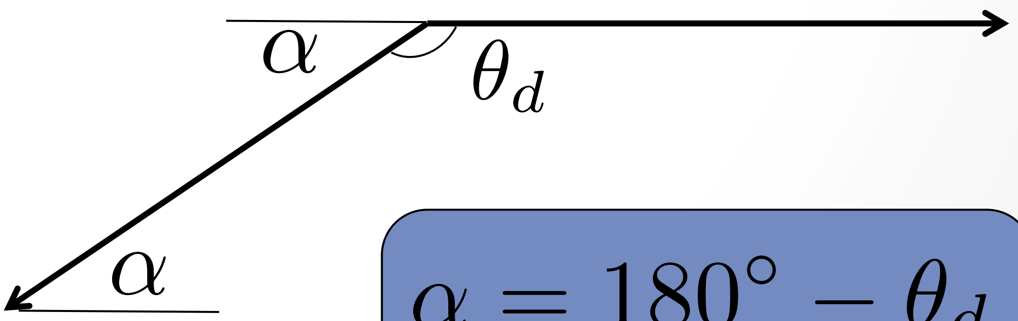
The *net deviation angle,* θ_d*, is*

$$\theta_d = 180^\circ + 2\theta_i - 4\theta_t$$

or

$$\theta_d(\theta_i) = 180^\circ + 2\theta_i - 4\sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$

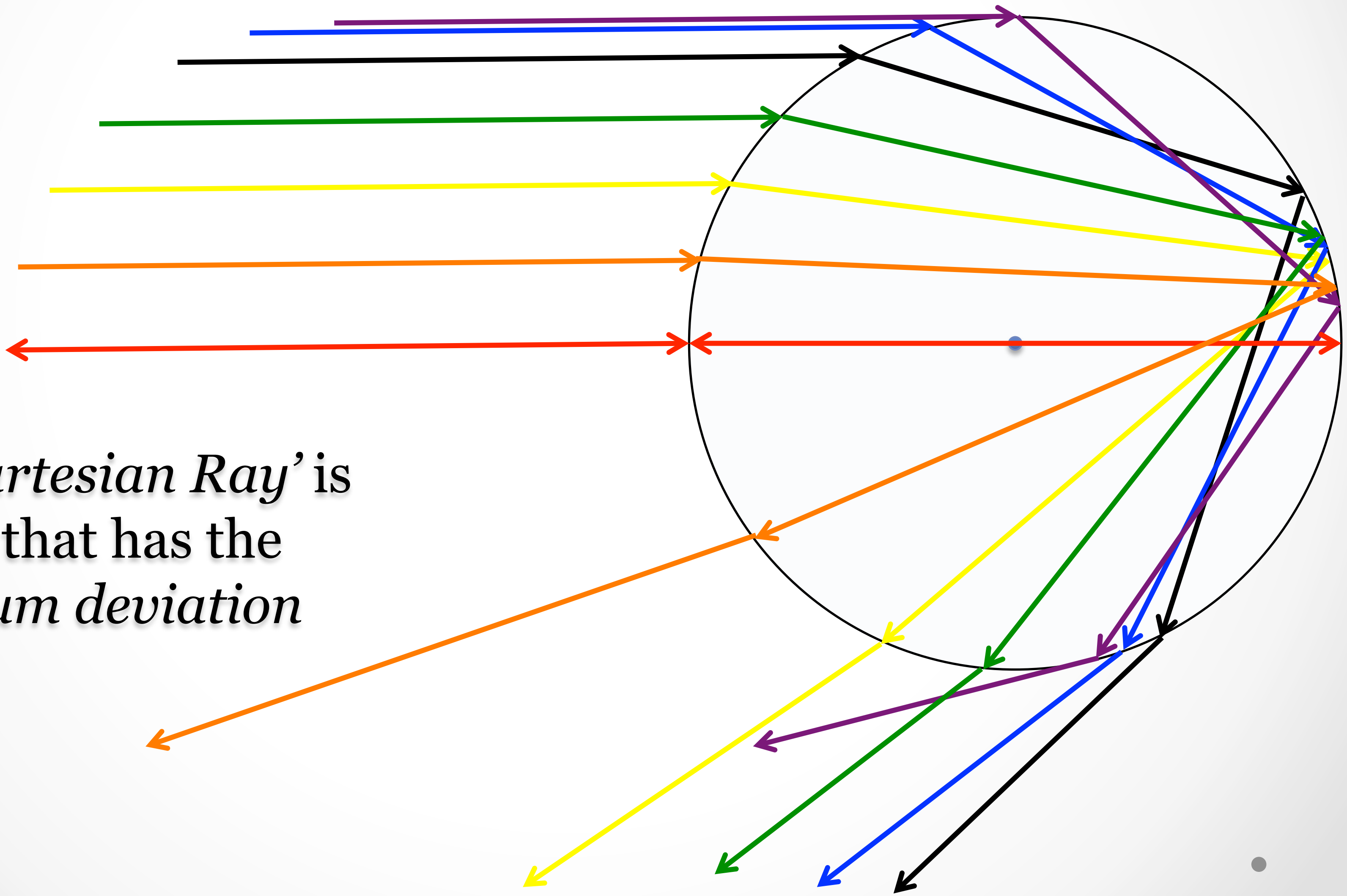
where we used Snell's Law.



$$\alpha = 180^\circ - \theta_d$$

↖ observed angle

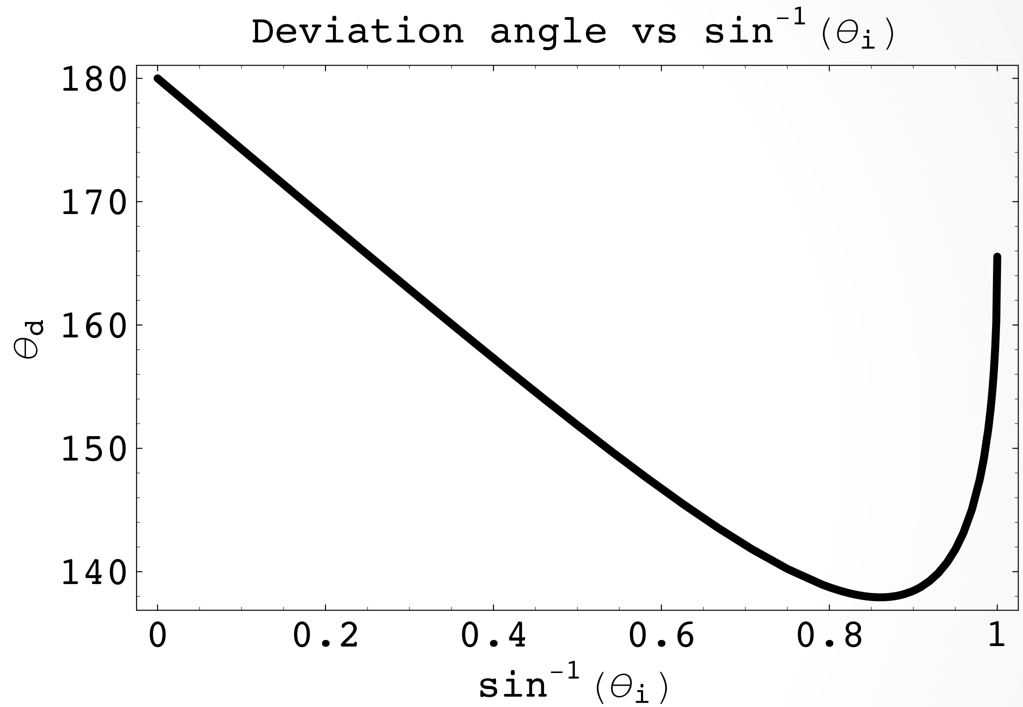
An infinite number of parallel sunbeams hit the spherical raindrop, so which ones do we see?



Notice:
The '*Cartesian Ray*' is the ray that has the *minimum deviation angle.*

Plot of θ_d vs $\sin^{-1}\theta_i$…

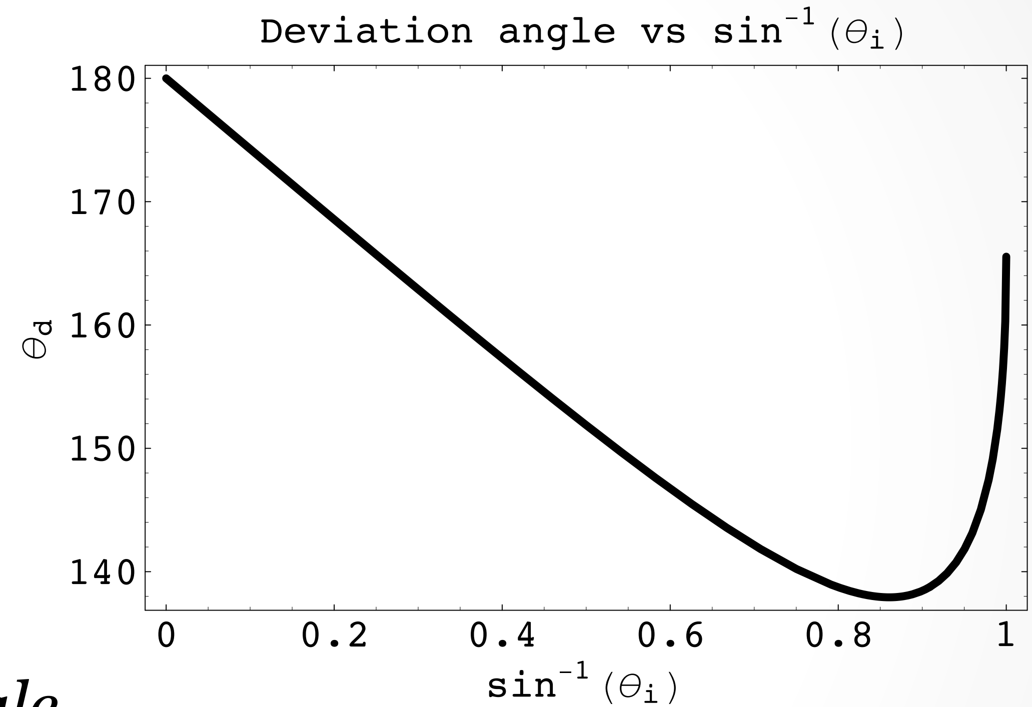
The backscattered rays cluster at the *minimum deviation angle*, yielding an enhanced brightness



The *net deviation angle is…*

$$\theta_d = 180^\circ + 2\theta_i - 4\sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$

Minimum deviation angle…



To calculate the
minimum deviation angle…

$$\frac{d\theta_d}{d\theta_i} = 0 \qquad \text{yields}$$

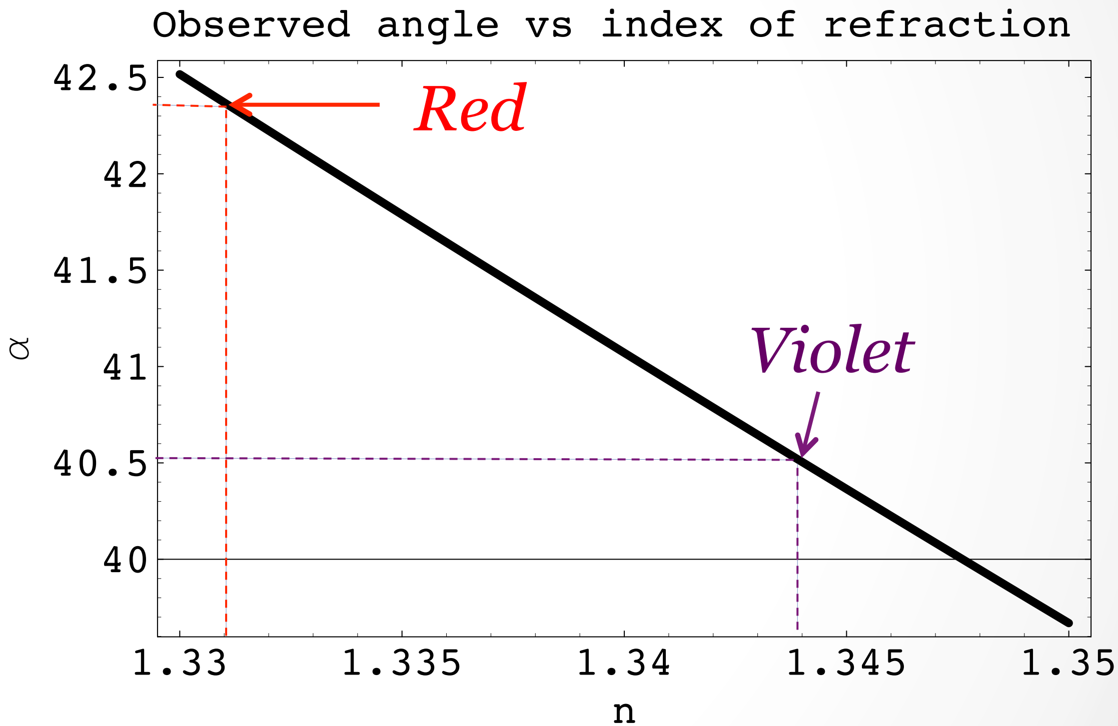
$$\sin\theta_i = \sqrt{\frac{4-n^2}{3}}$$

Dispersion...

Different frequencies of light have *different indices of refraction.*

$$n_{t,r} = 1.331$$
$$n_{t,v} = 1.344$$

$$\alpha_r = 42.4^\circ$$
$$\alpha_v = 40.5^\circ$$
$$\Delta\alpha = 1.9^\circ$$

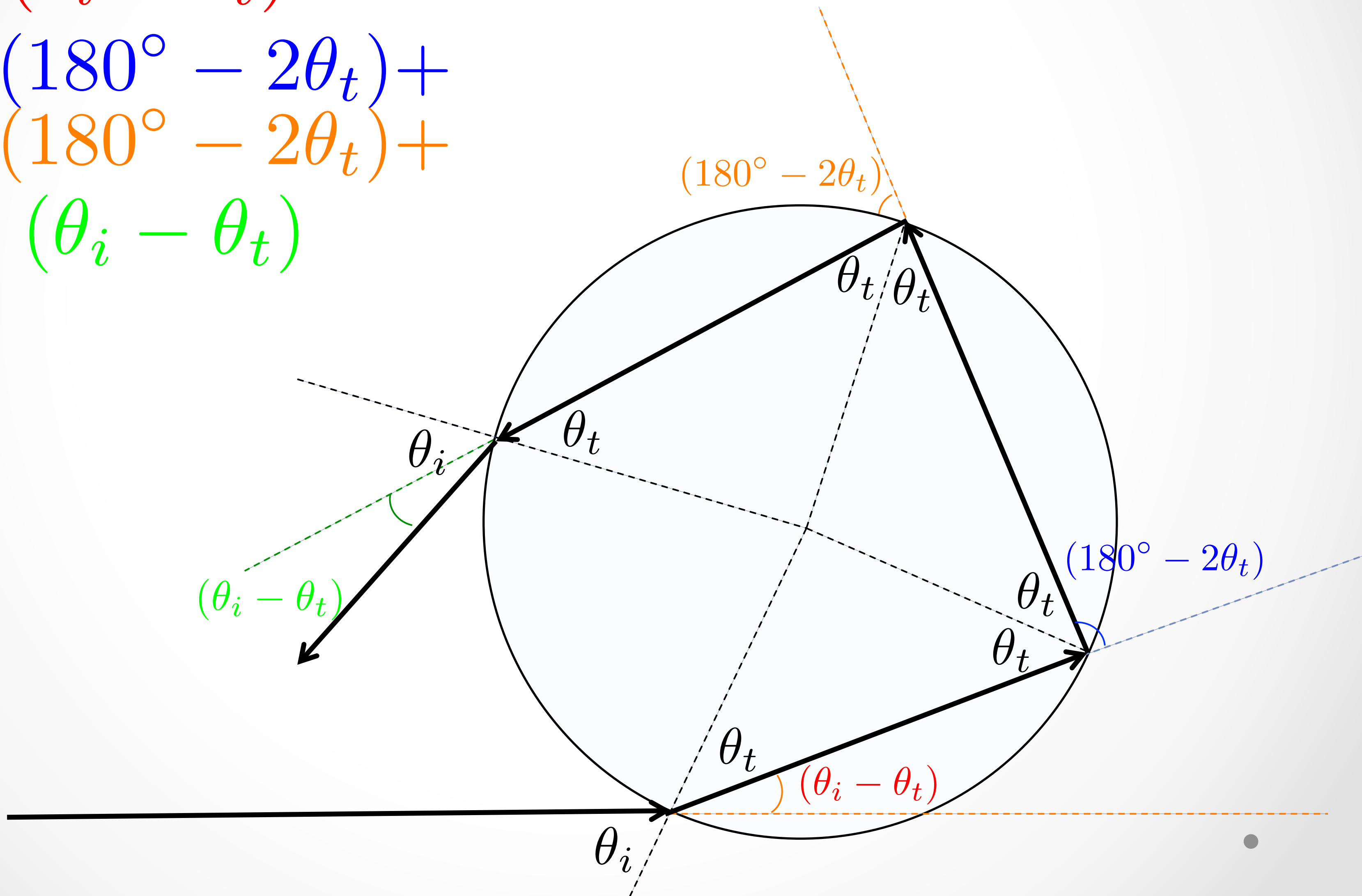


$$\alpha(n) = -2\sin^{-1}\left(\sqrt{\frac{4-n^2}{3}}\right) + 4\sin^{-1}\left(\sqrt{\frac{4-n^2}{3n^2}}\right)$$

Notice:
The 'outside' of the *primary rainbow*
- is **red**, whereas the 'inside' is **violet**!

Secondary Rainbow...

$$\theta_d = (\theta_i - \theta_t) + (180° - 2\theta_t) + (180° - 2\theta_t) + (\theta_i - \theta_t)$$



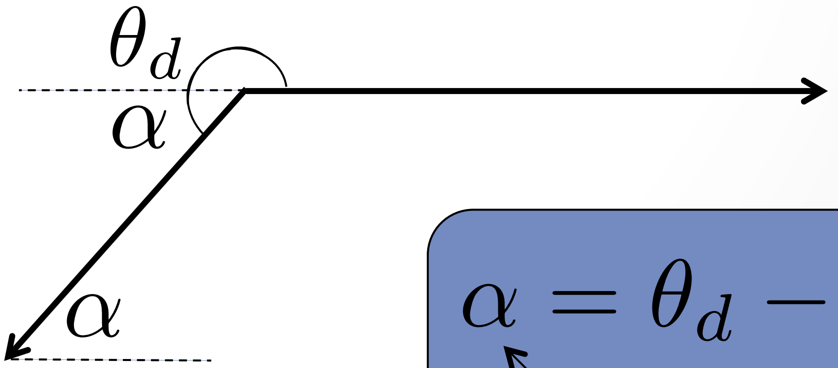
The *net deviation angle,* θ_d*, is…*

$$\theta_d = 360^\circ + 2\theta_i - 6\theta_t$$

or

$$\theta_d(\theta_i) = 360^\circ + 2\theta_i - 6\sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$

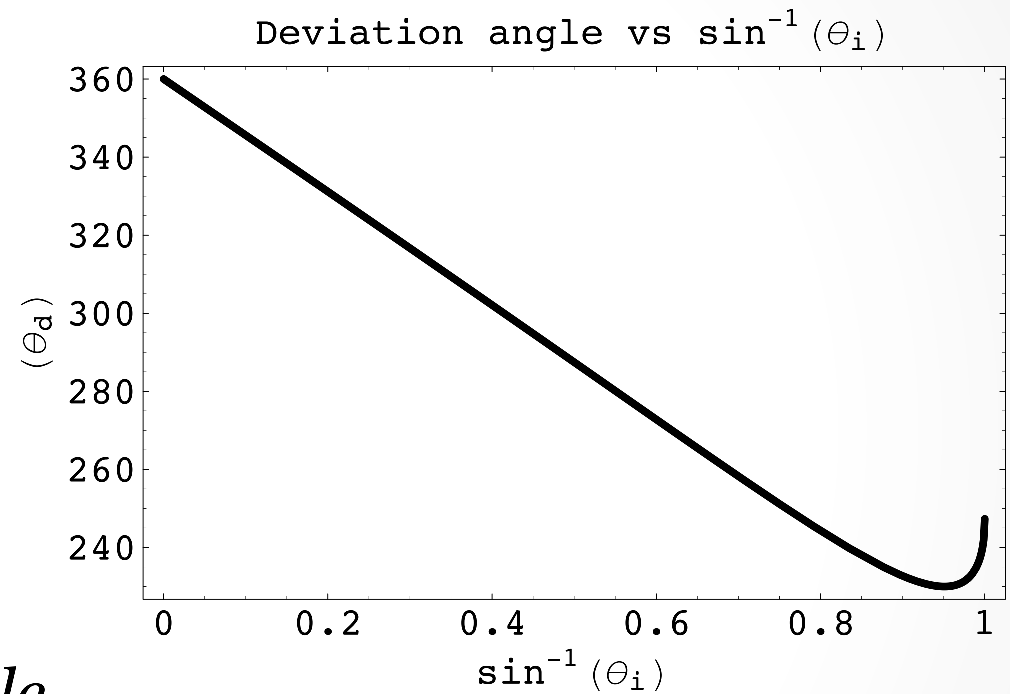
where we again used Snell's Law.



$$\alpha = \theta_d - 180^\circ$$

observed angle

Minimum deviation angle…



To calculate the *minimum deviation angle…*

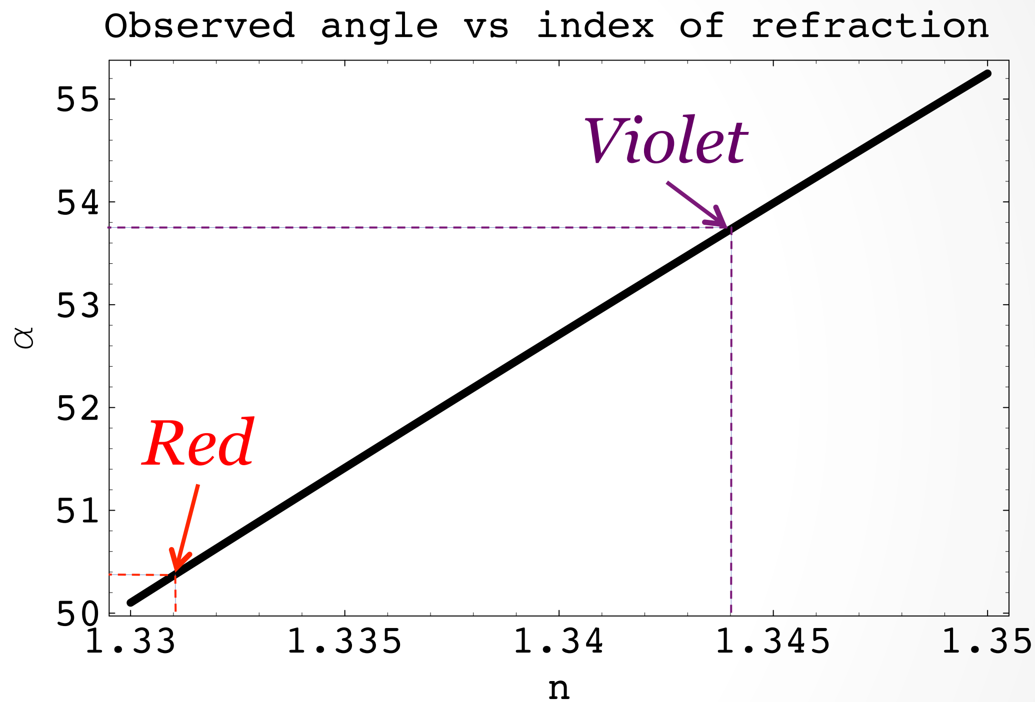
$$\frac{d\theta_d}{d\theta_i} = 0 \qquad \text{yields}$$

$$\sin\theta_i = \sqrt{\frac{9-n^2}{8}}$$

Secondary Rainbow…

$$n_{t,r} = 1.331$$
$$n_{t,v} = 1.344$$

$$\alpha_r = 50.3^\circ$$
$$\alpha_v = 53.8^\circ$$
$$\Delta\alpha = 3.5^\circ$$



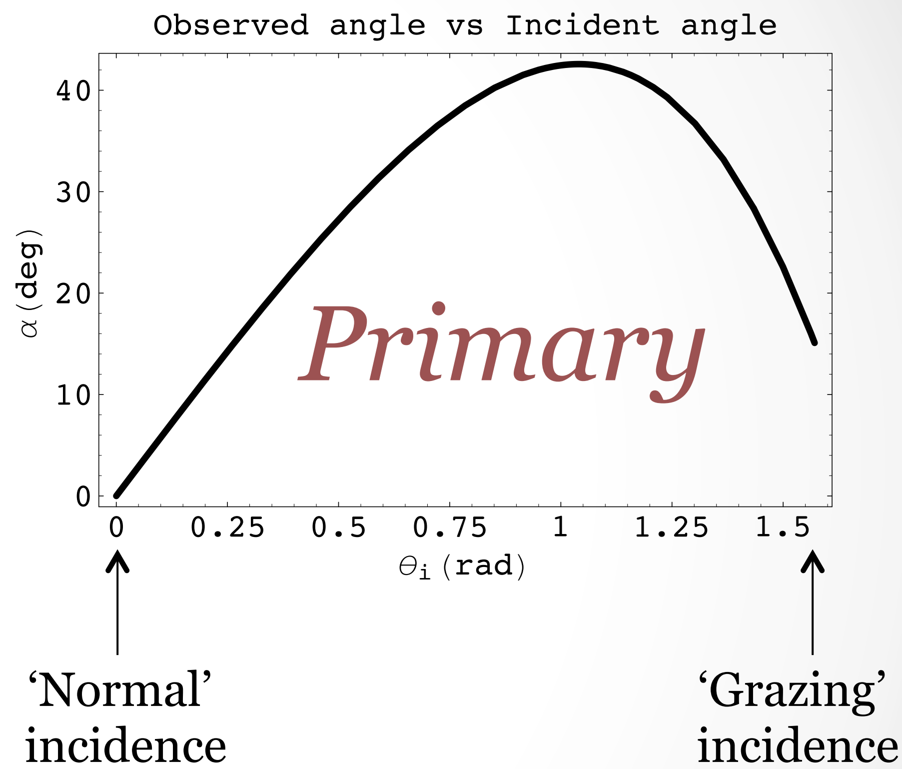
$$\alpha(n) = 180^\circ + 2\sin^{-1}\left(\sqrt{\frac{9-n^2}{8}}\right) - 6\sin^{-1}\left(\sqrt{\frac{9-n^2}{8n^2}}\right)$$

Notice:
The 'outside' of the *secondary rainbow* is **violet**, whereas the 'inside' is **red**!

Why is the interior region of the *primary* rainbow *bright*?

For the *primary rainbow*...
one has backscattering for *all* angles in the regime:

$$0 \leq \alpha \leq 42.4^\circ$$



Your eye receives backscattered light of *all wavelengths* from the *interior* of the primary rainbow
=> *Bright white light!*

Why is the region between the *primary and secondary* rainbows *dark*?

For the *primary rainbow…*
one has backscattering when

$$0 \leq \alpha \leq 42.4^\circ$$

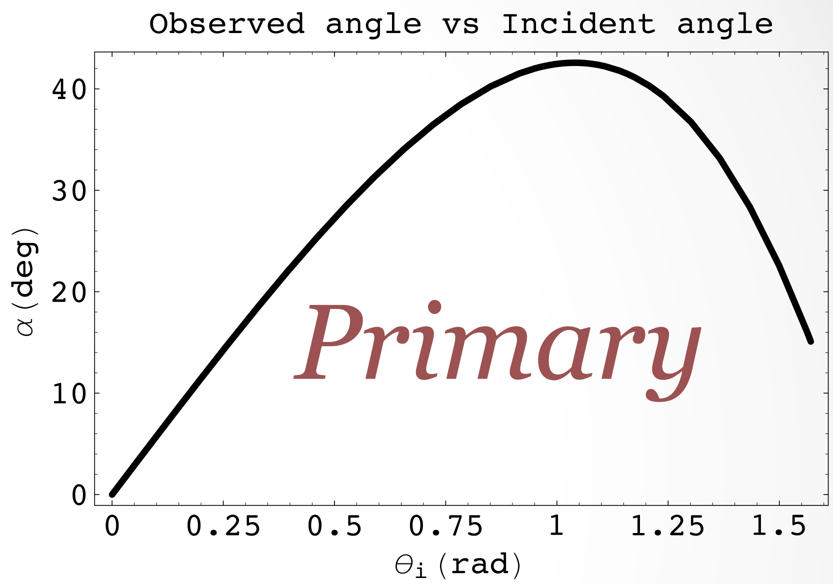
For the *secondary rainbow…*
one has backscattering when

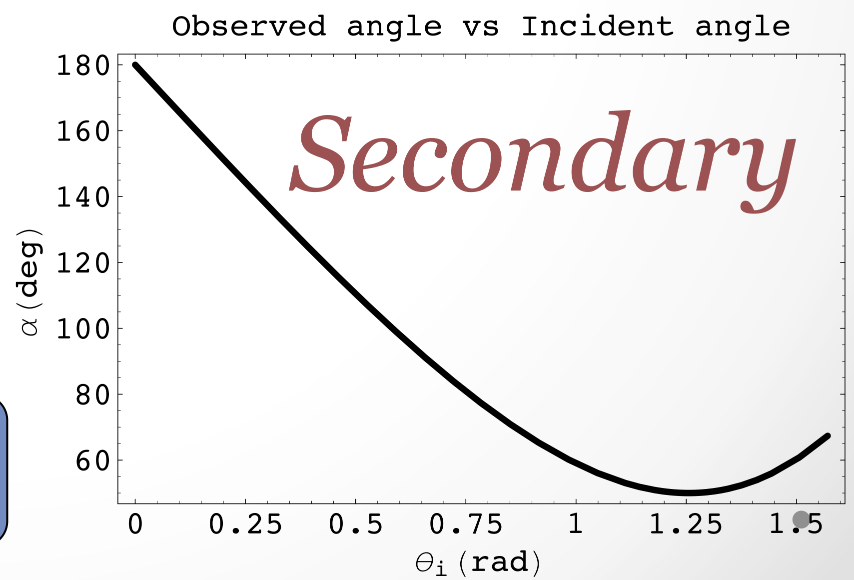
$$50.3^\circ \leq \alpha \leq 180^\circ$$

One has *ZERO* scattering from *one or two reflections* when

$$42.4^\circ < \alpha < 50.3^\circ$$

=> *Alexander's dark band!*





Is a 3rd (or *l* th) rainbow theoretically possible?

After allowing for *l internal reflections*,
the ***net deviation angle*** is…

$$\theta_d(\theta_i) = l(180^\circ) + 2\theta_i - 2(l+1)\sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$

To calculate the *minimum deviation angle*…

$$\frac{d\theta_d}{d\theta_i} = 0 \quad \text{yields} \quad \sin\theta_i = \sqrt{\frac{(l+1)^2 - n^2}{l(l+2)}}$$

⇒ The first 13 rainbows of water have been observed from a drop suspended in a spectrometer!*

*Jearl D. Walker, "Multiple rainbows from single drops of water and other liquids", *American Journal of Physics* Vol. 44, No. 5, May 1976.

Why are two rainbows sometimes visible in the sky, but one never sees a third (or fourth)?

For the *tertiary rainbow* (l = 3)…

$$\alpha_r = 137.5^\circ$$
$$\alpha_v = 144.3^\circ$$
$$\Delta\alpha = 6.8^\circ$$

So why don't you see it?

1. Each successive *Cartesian ray* is at a *greater* incident angle, therefore a reduction in intercepting *cross-sectional area*.
2. Larger $\Delta\alpha$ for each successive rainbow.
3. Loss of light due at each successive *reflection*.