Lecture Series on...

General Relativity and Differential Geometry

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<u>OUTLINE</u>

- Distance in 4D *Curved* Space
- Christoffel Symbols
- Einstein Field Equations
- Ricci Tensor & Ricci Scalar
- Schwarzschild Solution

Line element using *curved* space

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- $g_{\mu\nu}$ is the metric tensor
- $\eta_{\mu\nu}$ is the *flat* limit of $g_{\mu\nu}$
- $g_{\mu\nu} = g_{\nu\mu}$

- Again, repeated index $\equiv$ sum over index

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$

$$\begin{aligned} ds^2 &= g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + g_{02} dx^0 dx^2 + ... \\ &\quad + ... + g_{13} dx^1 dx^3 + ... + g_{33} dx^3 dx^3 \end{aligned} \tag{1}$$

- $g_{\mu\nu}$ defines the geometry of spacetime
- *Know* $g_{\mu\nu}$, *Know Geometry*

Example:

What is the metric tensor of the unit 2-sphere?

$$ds_2^2 \equiv d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$$

The metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{2}$$

writing the components of the metric...

$$g_{\theta\theta} = 1\ ,\ g_{\theta\phi} = g_{\phi\theta} = 0\ ,\ g_{\phi\phi} = \sin^2\theta$$

What is the inverse metric? $(g \cdot g^{-1} = 1)$

$$(g_{\mu\nu})^{-1} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}$$

with components ...

$$g^{\theta\theta} = 1\ ,\ g^{\theta\phi} = g^{\phi\theta} = 0\ ,\ g^{\phi\phi} = 1/\sin^2\theta$$

- All the ways curvature manifests itself rely on something called a "connection".

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma} \right)$$

where

$$\partial_\beta \equiv \frac{\partial}{\partial x^\beta}$$

- δ is a repeated index $\rightarrow$ Sum!

- Looks like a tensor... but it's not a tensor.

Notice:

$$\Gamma = \Gamma(g, \partial g)$$

- Christoffel is a function of the metric and the derivative of the metric

Example:

What are the possible connections for the unit 2-sphere?

$$d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$$

Components of the metric:

$$g_{\theta\theta} = 1\ ,\ g_{\theta\phi} = g_{\phi\theta} = 0\ ,\ g_{\phi\phi} = \sin^2\theta$$

Answer:

$$\Gamma^{\theta}_{\theta\theta}\ ,\ \Gamma^{\theta}_{\theta\phi}\ ,\ \Gamma^{\phi}_{\theta\theta}\ ,\ \Gamma^{\theta}_{\phi\phi}\ ,\ \Gamma^{\phi}_{\theta\phi}\ ,\ \Gamma^{\phi}_{\phi\phi}$$

What are the connections for the unit 2-sphere?

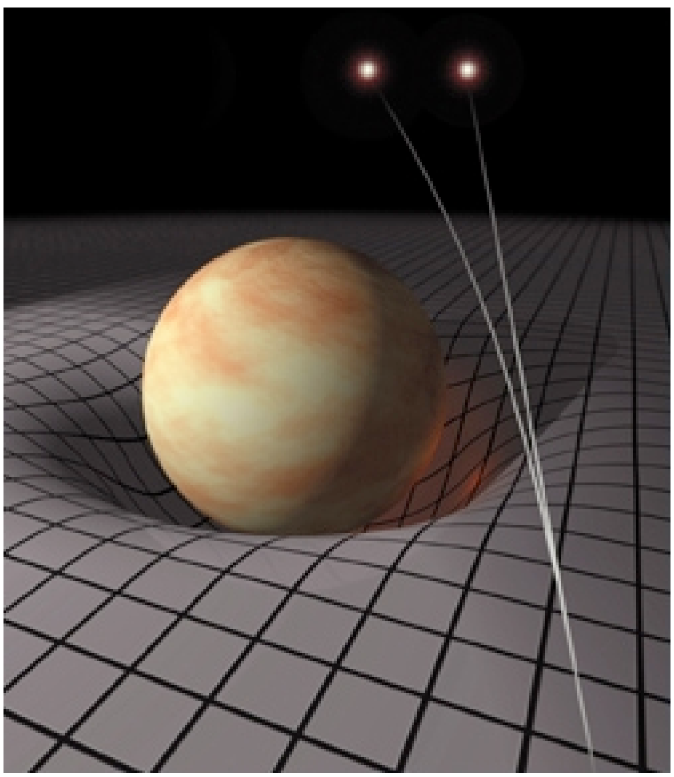
$$\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= \frac{1}{2} g^{\theta\delta}\left(\partial_\phi g_{\phi\delta} + \partial_\phi g_{\phi\delta} - \partial_\delta g_{\phi\phi}\right) \\ &= \frac{1}{2} g^{\theta\theta}\left(2\partial_\phi g_{\phi\theta} - \partial_\theta g_{\phi\phi}\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial\theta}(\sin^2\theta) = -\sin\theta\cos\theta \end{aligned} \tag{3}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta\ ,\ \Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\phi} = \Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi} = 0$$

General Theory of Relativity

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor describing the *curvature* of space.
- $T_{\mu\nu}$ is the stress-energy tensor which describes *matter*.



- *Matter tells space how to curve*
- *Space tells matter how to move*

Einstein Field Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

- Subscripts label elements of each matrix
- Set of 10 second-order, non-linear partial differential equations
- EM, Strong, & Weak $\rightarrow$ fields on spacetime

  $\Rightarrow$ Gravity is curvature of spacetime itself!

Einstein Field Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

where

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\beta\nu}$$

is the *Ricci Tensor* and

$$R = g^{\mu\nu} R_{\mu\nu}$$

is the *Ricci Scalar*.

Notice:

$$R_{\mu\nu} = R_{\mu\nu}(\Gamma, \partial\Gamma) \ \text{ but } \ \Gamma = \Gamma(g, \partial g)$$

The Einstein Tensor

$$G_{\mu\nu} = G_{\mu\nu}(g, \partial g, \partial^2 g)$$

is written entirely in terms of the metric tensor!

Example: Line element for 2-sphere of *radius* a

$$ds_2^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^{\phi}_{\theta\phi} &= \cot\theta \\
0 &= \Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\phi} = \Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi}
\end{aligned} \quad (4)$$

What are the Ricci Tensors?

$$R_{\phi\phi} = \partial_\alpha \Gamma^{\alpha}_{\phi\phi} - \partial_\phi \Gamma^{\alpha}_{\phi\alpha} + \Gamma^{\alpha}_{\phi\phi}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\phi\alpha}\Gamma^{\alpha}_{\beta\phi}$$

· Summing out indicies...

$$\begin{aligned}
R_{\phi\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\beta}_{\theta\beta} - \Gamma^{\beta}_{\phi\theta}\Gamma^{\theta}_{\beta\phi} - \Gamma^{\beta}_{\phi\phi}\Gamma^{\phi}_{\beta\phi} \\
&= \partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} \\
&= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}
\end{aligned} \quad (5)$$

- Plugging in the functions

$$\begin{aligned} R_{\phi\phi} &= \partial_\theta(-\sin\theta\cos\theta) + \cot\cdot\theta\sin\theta\cos\theta \\ &= \sin^2\theta \end{aligned} \tag{6}$$

- Likewise,

$$R_{\theta\theta} = 1$$

What is the Ricci Scalar?

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= \frac{1}{a^2\sin^2\theta}\cdot\sin^2\theta + \frac{1}{a^2} \\ &= \frac{2}{a^2} \end{aligned} \tag{7}$$

The Ricci Scalar...

- ... completely characterizes curvature (2D)
- ... is constant on 2-sphere
- ... decreases with increasing a

Einstein Field Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Method of Solving the EFE's:

- Choose mass/energy source
- Choose a metric *ansatz*
- Plug metric *ansatz* into the EFE's
- Obtain and solve coupled, nonlinear PDE's!!

In 1916, Karl Schwarzschild presents an EXACT solution!

- Consider a spherically symmetric, *static* source

The vacuum equations are...

$$R_{\mu\nu} = 0$$

- Choose metric *ansatz*

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega^2$$

- Plug metric ansatz into the EFE's

The Christoffel Symbols are...

$$\begin{aligned} \Gamma^t_{tr} &= \partial_r A & \Gamma^r_{tt} &= e^{2(A-B)}\partial_r A & \Gamma^r_{rr} &= \partial_r B \\ \Gamma^\theta_{r\theta} &= \frac{1}{r} & \Gamma^r_{\theta\theta} &= -re^{-2B} & \Gamma^\phi_{r\phi} &= \frac{1}{r} \\ \Gamma^r_{\phi\phi} &= -re^{-2B}\sin^2\theta & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta & \Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta} \end{aligned} \tag{8}$$

The vacuum equations are...

$$
\begin{aligned}
R_{tt} &= e^{2(A-B)}\left[\partial_r^2 A + (\partial_r A)^2 - \partial_r A \partial_r B + \frac{2}{r}\partial_r A\right] = 0 \\
R_{rr} &= -\partial_r^2 A - (\partial_r A)^2 + \partial_r A \partial_r B + \frac{2}{r}\partial_r B = 0 \\
R_{\theta\theta} &= e^{-2B}\left[r(\partial_r B - \partial_r A) - 1\right] + 1 = 0 \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} = 0 \qquad (9)
\end{aligned}
$$

Since R_{tt} and R_{rr} vanish independently, we can write

$$
\begin{aligned}
e^{2(B-A)} R_{tt} + R_{rr} = 0 \\
\frac{2}{r}(\partial_r A + \partial_r B) &= 0 \qquad (10)
\end{aligned}
$$

which yields

$$A = -B$$

Now $R_{\theta\theta} = 0$ implies

$$
\begin{aligned}
e^{2A}(2r\partial_r A + 1) &= 1 \\
\partial_r(re^{2A}) = 1 \quad \rightarrow \quad e^{2A} &= \left(1 - \frac{R_S}{r}\right) \qquad (11)
\end{aligned}
$$

The Schwarzschild Solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the 2-sphere.

Notice:

- When $r \to R_S = 2GM$

  $\Rightarrow \; g_{tt} \to 0 \; \& \; g_{rr} \to \infty$

  - $R_S = 2GM$ is the Schwarzschild radius.

    $\Rightarrow$ Not a real singularity

- When $r \to 0$

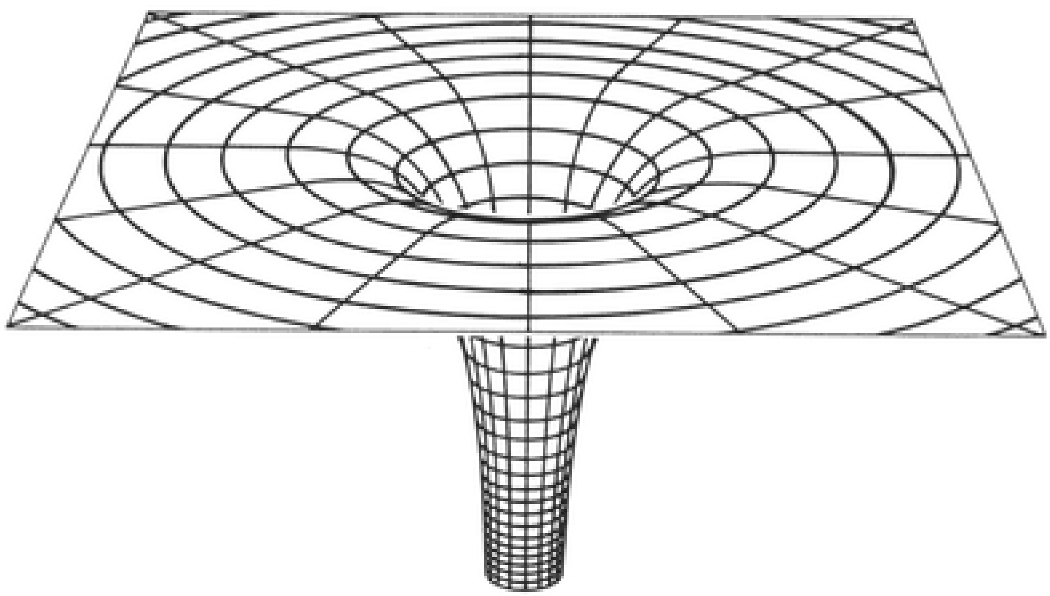
  $\Rightarrow \; g_{tt} \to \infty \; \& \; g_{rr} \to 0$

  This is a spacetime singularity!

- Schwarzschild line-element predicts Black holes!

The Schwarzschild Solution

$$ds^2_{sch} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2$$



Notice:

$$\begin{aligned} \lim_{M\to 0} ds^2_{sch} &\to -dt^2 + dr^2 + r^2 d\Omega^2 \\ \lim_{r\to\infty} ds^2_{sch} &\to -dt^2 + dr^2 + r^2 d\Omega^2 \text{ - asymptotic flatness} \end{aligned} \qquad (12)$$

Consider *radial null curves* (θ, ϕ constants)

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

This yields

$$\frac{dt}{dr} = \pm \frac{1}{\left(1 - \frac{2GM}{r}\right)}$$

Notice:

$$\lim_{r \to \infty} \frac{dt}{dr} \to \pm 1$$

$$\lim_{r \to 2GM} \frac{dt}{dr} \to \pm \infty$$

- The "light cone" closes up!

  $\Rightarrow$ Distant observer never sees the light ray get there.