Elliptical-like orbits on a warped spandex fabric

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Kepler's 3 Laws of planetary motion

Kepler's 1st Law..

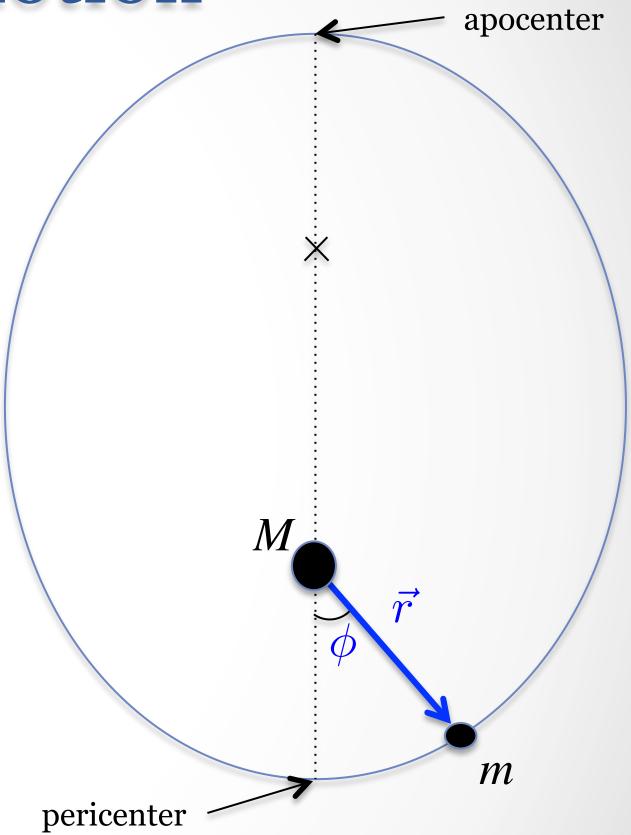
- The planets move in *elliptical* orbits with the sun at one focus.

Kepler's 2nd Law..

- A line extending from the Sun to any planet sweeps out *equal areas in equal times*.

Kepler's 3rd Law..

$$T^2 = \left(\frac{4\pi^2}{G}\right) \cdot \frac{r^3}{M}$$



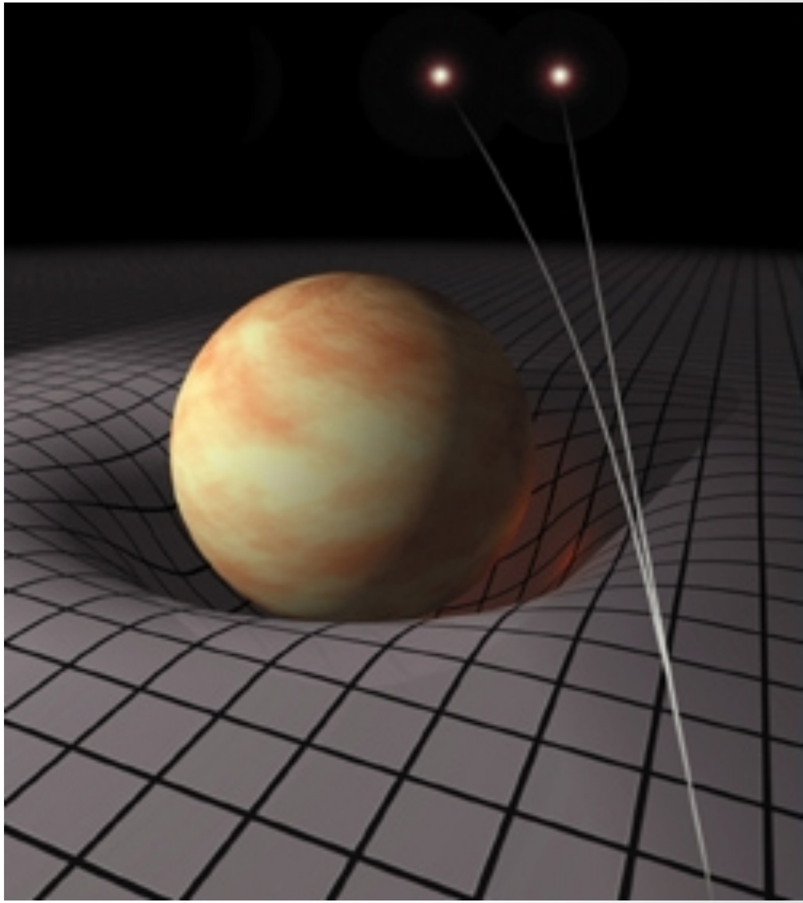
$$r(\phi) = \frac{r_0}{1 + \varepsilon \cos\phi}$$

Einstein's theory of general relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $G_{\mu\nu}$ describes the *curvature of spacetime*
- $T_{\mu\nu}$ describes the *matter & energy in spacetime*

Matter tells *space* how to *curve*, *space* tells *matter* how to *move*.



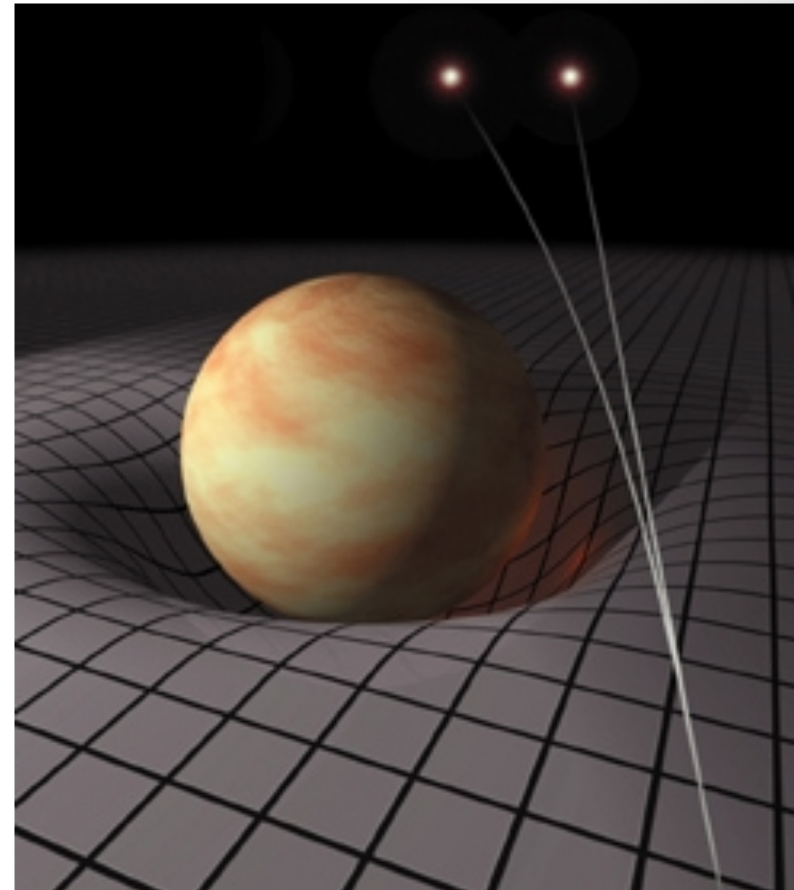
Sean M. Carrol, *Spacetime and Geometry*: *An Introduction to Einstein's General Relativity* (Addison Wesley, 2004)

The physics of the analogy…

Is there a 2D surface that will generate orbits that obey Kepler's 3 laws?

For a marble orbiting on a warped elastic fabric, how does the *period* of the orbit relate to the *radial distance*?

Can one generate *elliptical-like orbits* on the warped elastic surface?



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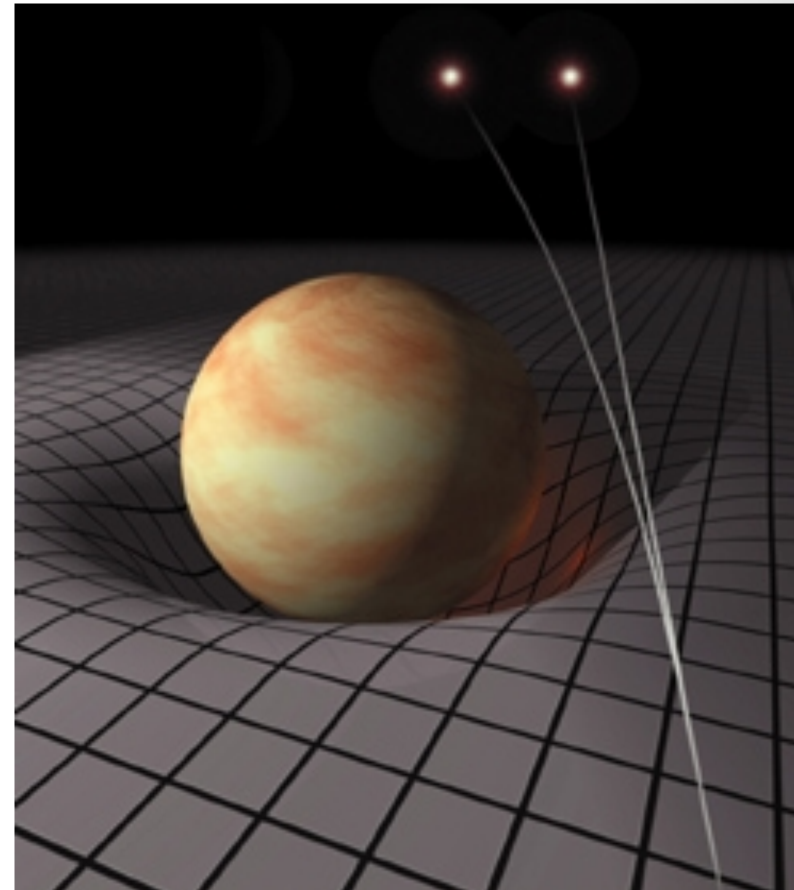

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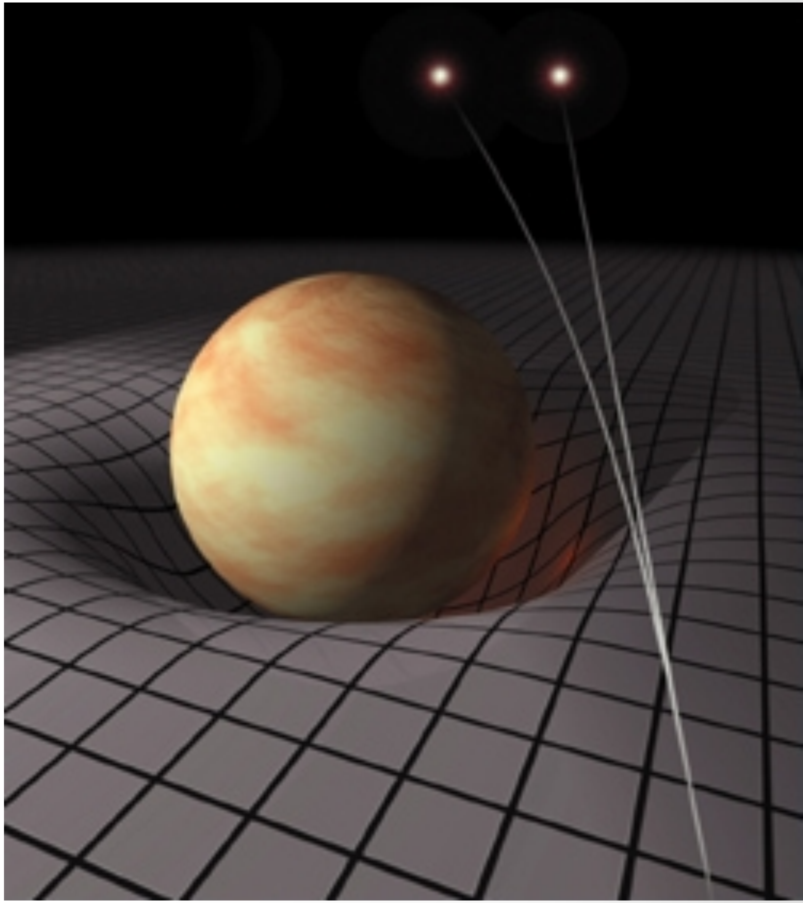

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Can one generate *elliptical-like orbits* on the warped elastic surface?

-> Yes, stay tuned!

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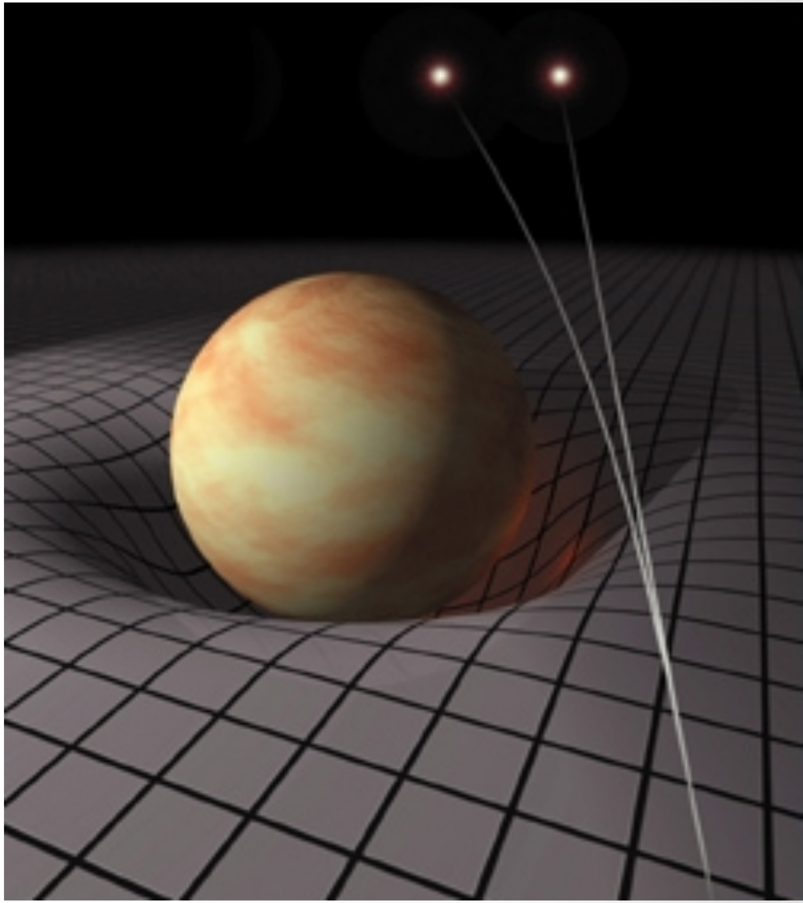

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Outline

- A marble rolling on a cylindrically symmetric surface
  (Lagrangian dynamics)
- The shape of the spandex fabric (Calculus of Variations)
- Small slope regime
  - Angular separation between successive apsides
  - Experiment
- Large slope regime
  - Angular separation between successive apsides
  - Experiment
- Elliptical-like orbits in GR

A marble rolling on a cylindrically symmetric surface

- is described by a Lagrangian of the form..

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}I\omega^2 - mgz$$

- Now, for the rolling marble..

$$I = \frac{2}{5}mR^2 \text{ and } \omega^2 = v^2/R^2 \qquad \text{so} \qquad \frac{1}{2}I\omega^2 = \frac{1}{5}mv^2$$

Notice:

- The marble is constrained to reside on the fabric..
- $$z = z(r)$$

The Lagrange equations of motion

- take the form...

$$* \quad (1+z'^2)\ddot{r} + z'z''\dot{r}^2 - r\dot{\phi}^2 + \frac{5}{7}gz' = 0$$

$$\dot{\phi} = \frac{5\ell}{7r^2}$$

- define the differential operator...

$$\frac{d}{dt} = \frac{5\ell}{7r^2}\frac{d}{d\phi}$$

- * becomes...

$$(1+z'^2)\frac{d^2r}{d\phi^2} + \left(z'z'' - \frac{2}{r}(1+z'^2)\right)\left(\frac{dr}{d\phi}\right)^2 - r + \frac{7g}{5\ell^2}\cdot z'r^4 = 0$$

The equation of motion for a rolling marble on a cylindrically symmetric surface…

$$(1+z'^2)\frac{d^2r}{d\phi^2}+\left(z'z''-\frac{2}{r}(1+z'^2)\right)\left(\frac{dr}{d\phi}\right)^2-r+\frac{7g}{5\ell^2}\cdot z'r^4=0$$

For elliptical-like orbits with small eccentricities…

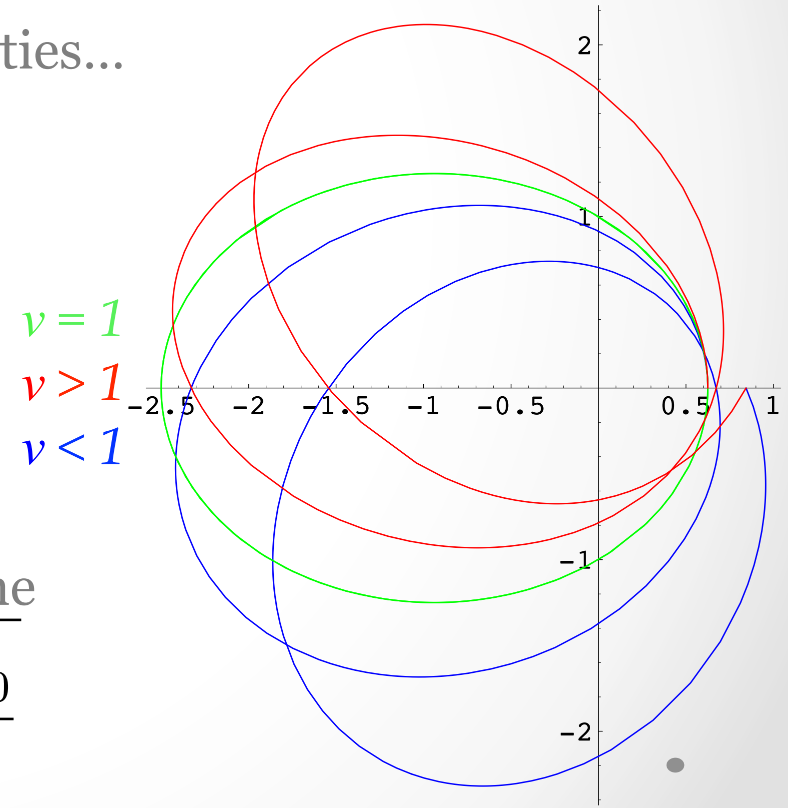
$$r(\phi)=r_0(1-\varepsilon\cos(\nu\phi))$$

- where ν is the *precession parameter*

$$\nu\equiv\frac{360^\circ}{\Delta\phi}$$

- Inserting the approximate solution into the equation of motion yields

$$\nu=\sqrt{\frac{3z_0'+z_0''r_0}{z_0'+z_0'^3}}$$



The slope of the spandex fabric

Technique:

1. Construct potential energy (PE) *integral functional* of spandex fabric.
   - *i.* *Elastic* PE of the *spandex*.
   - *ii.* *Gravitational* PE of the *spandex*.
   - *iii.* *Gravitational* PE of the *central mass*.

2. Apply *Calculus of Variations*.

   ⇒ The elastic fabric-mass system will assume the shape which *minimizes* the *total* PE of the system.

i. and *iii*. first considered in "Comment on "The shape of 'the Spandex' and orbits upon its surface", Am. J. Phys. 70 (10), 1056-1058 (2002)

The slope of the spandex fabric

The Euler-Lagrange equation can be integrated once and takes the form..

$$rz'\left[1-\frac{1}{\sqrt{1+z'^2}}\right]=\alpha(M+\sigma_0\pi r^2)$$

- where we defined the parameter..

$$\alpha\equiv\frac{g}{2\pi E}$$

The slope of the spandex fabric

The Euler-Lagrange equation can be integrated once and takes the form..

$$rz'\left[1-\frac{1}{\sqrt{1+z'^2}}\right]=\alpha(M+\sigma_0\pi r^2)$$

mass of central object

areal mass density

- where we defined the parameter..

$$\alpha\equiv\frac{g}{2\pi E}$$

Modulus of elasticity

The angular separation & the slope of the spandex fabric

$$\Delta\phi = 360^\circ \sqrt{\frac{z_0'(1+z_0'^2)}{3z_0' + r_0 z_0''}}$$

$$rz'\left[1 - \frac{1}{\sqrt{1+z'^2}}\right] = \alpha(M + \sigma_0 \pi r^2)$$

- Angular separation between successive like-apsides
- The slope of the spandex fabric

- Small slope regime..

$$z'(r) \ll 1 \quad \text{so} \quad \frac{1}{\sqrt{1+z'^2}} = 1 - \frac{1}{2}z'^2 + o(z'^4)$$

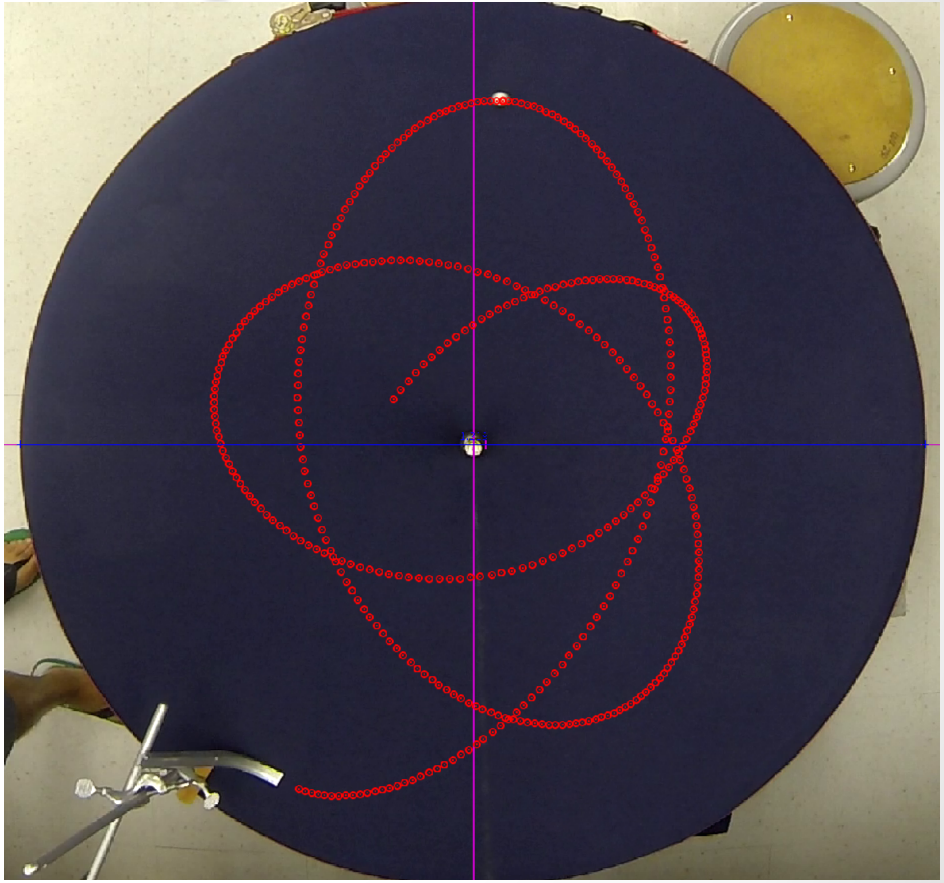
- Large slope regime..

$$z'(r) \gg 1 \quad \text{so} \quad \frac{1}{\sqrt{1+z'^2}} = \frac{1}{z'}\frac{1}{\sqrt{1+1/z'^2}} = \frac{1}{z'}(1 - \frac{1}{2z'^2} + o(1/z'^4))$$

Elliptical-like orbits on the spandex fabric

- 4 ft. diameter trampoline frame
  - styrofoam insert for *zero* pre-stretch
  - truck tie down around perimeter
- Camera mounted directly above, ramp mounted on frame.
- Position determined every 1/60 s and *average radius*, r_{ave}, calculated per $\Delta\phi$.
- $\Delta\phi$ from r_{max} to r_{max} can be measured.
- $\Delta\phi$ can be calculated via…

$$\Delta\phi = 360^\circ \sqrt{\frac{z_0'(1+z_0'^2)}{3z_0' + r_0 z_0''}}$$

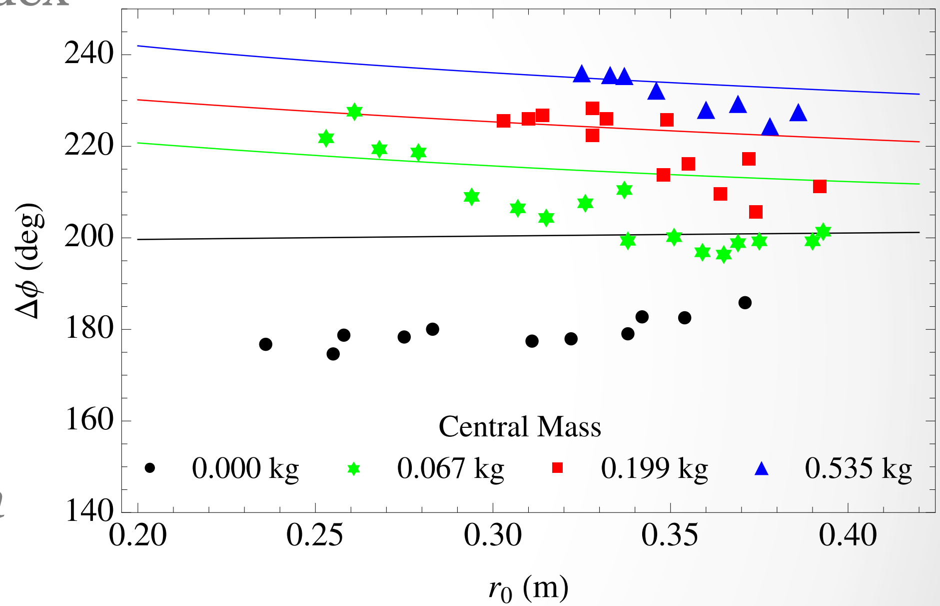


The three orbits imaged here were found to have angular separations between successive apocenters of $\Delta\phi = 213.5^\circ, 225.7^\circ, 223.9^\circ$ and eccentricities of $\varepsilon = 0.31,\ 0.29,\ 0.31$

Elliptical-like orbits in the small slope regime

- For $z'(r) \ll 1$, the slope of the spandex surface takes the form…

$$z'(r) \simeq \left(\frac{2\alpha}{r}\right)^{1/3} (M + \sigma_0 \pi r^2)^{1/3}$$



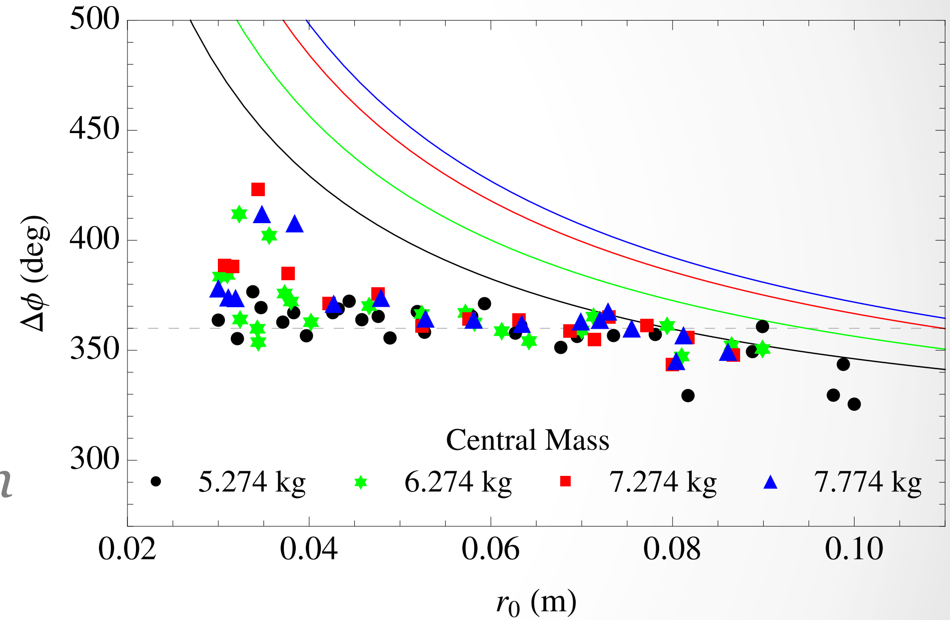
- Plugging this into the equation determining the *angular separation* yields a theoretical value of…

$$\Delta\phi = 220^\circ \left[1 + \left(2\alpha(M + \sigma_0\pi r^2)/r_0\right)^{2/3}\right]^{1/2} \left[1 + \frac{1}{4}\frac{\sigma_0 \pi r^2}{(M + \sigma_0 \pi r^2)}\right]^{-1/2}$$

Elliptical-like orbits in the large slope regime

- For $z'(r) \gg 1$, the slope of the spandex surface takes the form…

$$z'(r) \simeq 1 + \frac{\alpha M}{r}$$



- Plugging this into the equation determining the *angular separation* yields a theoretical value of…

$$\Delta\phi = 360^\circ \sqrt{\frac{(1+\alpha M/r_0)}{(3+2\alpha M/r_0)}\left[1+\left(1+\frac{\alpha M}{r_0}\right)^2\right]}$$

Elliptical-like orbits in the large slope regime

Notice:

- For a large central mass and a *very* small average radial distance, the angular separation yields a limiting behavior

$$\lim_{\alpha M/r_0 \gg 1} \Delta\phi \simeq \frac{360^\circ}{\sqrt{2}} \cdot \frac{\alpha M}{r_0}$$

- when $\alpha M/r_0 > \sqrt{2}$, $\Delta\phi > 360^\circ$!

Elliptical-like orbits with small eccentricities in GR

The equation of motion for an object of mass *m* orbiting about a *spherically symmetric* object of *mass M*, in the presence of a *cosmological constant* (or *vacuum energy*), Λ..

$$* \quad \ddot{r} + \frac{GM}{r^2} - \frac{\ell^2}{r^3} + \frac{3GM\ell^2}{c^2 r^4} - \frac{1}{3}\Lambda c^2 r = 0$$

$$\dot{\phi} = \frac{\ell}{r^2}$$

- define the differential operator..

$$\frac{d}{d\tau} = \frac{\ell}{r^2}\frac{d}{d\phi}$$

- * becomes..

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 + \frac{GM}{\ell^2}r^2 - r + \frac{3GM}{c^2} - \frac{\Lambda c^2}{3\ell^2}r^5 = 0$$

Elliptical-like orbits with small eccentricities in GR

The orbital equation of motion…

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 + \frac{GM}{\ell^2}r^2 - r + \frac{3GM}{c^2} - \frac{\Lambda c^2}{3\ell^2}r^5 = 0$$

For elliptical-like orbits with small eccentricities…

$$r(\phi) = r_0(1 - \varepsilon\cos(\nu\phi))$$

Precessing elliptical orbits in GR with small eccentricities

Case I: $\Lambda = 0$

- We find the solution, to 1st order in the eccentricity, when..

$$\ell^2 = GMr_0\left(1 - \frac{3GM}{c^2 r_0}\right)^{-1}$$

$$\nu^2 = 1 - \frac{6GM}{c^2 r_0}$$

Notice:

Innermost stable circular orbit

- When $r_0 < 6GM/c^2 \equiv r_{ISCO}$, ν becomes complex:
  elliptical-like orbits not allowed!
- When $r_0 < 3GM/c^2$, ν & ℓ become complex: *no circular orbits*.

Precessing elliptical orbits in GR with small eccentricities

Case I: $\Lambda = 0$

- The angular separation between successive apocenters takes the form...

$$^{*}\Delta\phi = 360^\circ \left(1 - \frac{r_{ISCO}}{r_0}\right)^{-1/2} \quad \text{where} \quad r_{ISCO} \equiv 6GM/c^2$$

- expand about...

$$r_0 \equiv r_{ISCO} + r \qquad \text{where} \qquad r \ll r_{ISCO}$$

- * becomes...

$$\lim_{r \ll 6GM/c^2} \Delta\phi \simeq 360^\circ \sqrt{6} \cdot \sqrt{\frac{GM/c^2}{r}}$$

Precessing elliptical orbits in GR with small eccentricities

Case I: Λ = 0, The behavior of the angular separation…

- in GR, near the *innermost stable circular orbit*…

$$\lim_{r \ll 6GM/c^2} \Delta\phi \simeq 360^\circ \sqrt{6} \cdot \sqrt{\frac{GM/c^2}{r}}$$

- of the marble, in the large slope regime…

$$\lim_{r_0 \ll \alpha M} \Delta\phi \simeq \frac{360^\circ}{\sqrt{2}} \cdot \frac{\alpha M}{r_0}$$

Notice:

- Both expressions *diverge* in the limit of vanishingly small distances.
- α plays the role of G/c^2; both set the scale of their respective theories.

Precessing elliptical orbits in GR with small eccentricities

Case II: $\Lambda \neq 0$

- We find the solution, to 1st order in the eccentricity, when..

$$\ell^2 = Gr_0\left(1 - \frac{3GM}{c^2 r_0}\right)^{-1}\left(M - 2\cdot\frac{4}{3}\pi r_0^3\cdot\rho_0\right)$$
$$\nu^2 = 1 - \frac{6GM}{c^2 r_0} - 6\left(1 - \frac{3GM}{c^2 r_0}\right)\frac{\frac{4}{3}\pi r_0^3\cdot\rho_0}{\left(M - 2\cdot\frac{4}{3}\pi r_0^3\cdot\rho_0\right)}$$

- The angular separation between successive apocenters takes the form…

$$\Delta\phi = 360^\circ\left(1 - \frac{6GM}{c^2 r_0}\right)^{-1/2}\left[1 - 6\left(\frac{1 - 3GM/c^2 r_0}{1 - 6GM/c^2 r_0}\right)\frac{\frac{4}{3}\pi r_0^3\cdot\rho_0}{\left(M - 2\cdot\frac{4}{3}\pi r_0^3\cdot\rho_0\right)}\right]^{-1/2}$$

Precessing elliptical orbits in GR with small eccentricities

Case II: $\Lambda \neq 0$, The behavior of the angular separation…

- in GR, about a *static, spherically symmetric massive object in the presence of a constant vacuum energy*..

$$\Delta\phi = 360^\circ \left(1 - \frac{6GM}{c^2 r_0}\right)^{-1/2} \left[1 - 6\left(\frac{1 - 3GM/c^2 r_0}{1 - 6GM/c^2 r_0}\right) \frac{\frac{4}{3}\pi r_0^3 \cdot \rho_0}{(M - 2 \cdot \frac{4}{3}\pi r_0^3 \cdot \rho_0)}\right]^{-1/2}$$

- of the marble, in the *small* slope regime…

$$\Delta\phi = 220^\circ \left[1 + \left(2\alpha(M + \sigma_0 \pi r^2)/r_0\right)^{2/3}\right]^{1/2} \left[1 + \frac{1}{4} \frac{\sigma_0 \pi r^2}{(M + \sigma_0 \pi r^2)}\right]^{-1/2}$$

Notice:

- The areal mass density, σ_o, of the spandex fabric plays the role of a *negative* vacuum energy density, $-\rho_o$, of spacetime.

Conclusion

- We find good agreement between theory and experiment for the *angular separation between successive apsides*, $\Delta\phi$.

- In the large slope regime, $\Delta\phi > 360^\circ$ for small radii and large central mass.

- $\Delta\phi$ *diverges* in the limit of *vanishing small distances* for
  - the marble in the large slope regime.
  - a particle near the *innermost stable circular orbit*.

- The areal mass density, σ_o, of the spandex fabric plays the role of a *negative* vacuum energy density, $-\rho_o$, of spacetime.

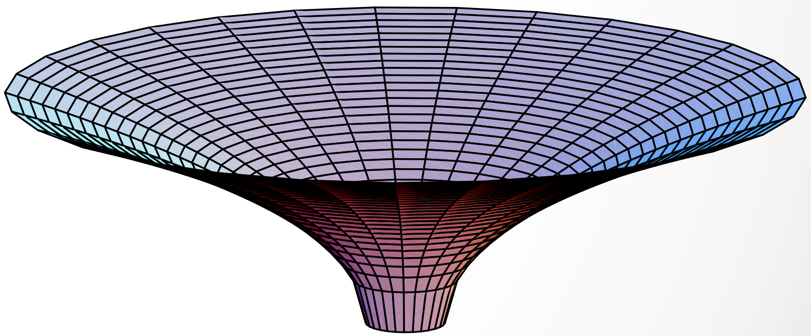
Einstein's theory of general relativity

Consider a *static, spherically symmetric, massive object…*

Embedding diagram ($t = t_0$, $\theta = \pi/2$)..

- 2D equatorial 'slice' of the 3D space at one moment in time

$$z(r) = 2\sqrt{\frac{2GM}{c^2}\left(r - \frac{2GM}{c^2}\right)}$$



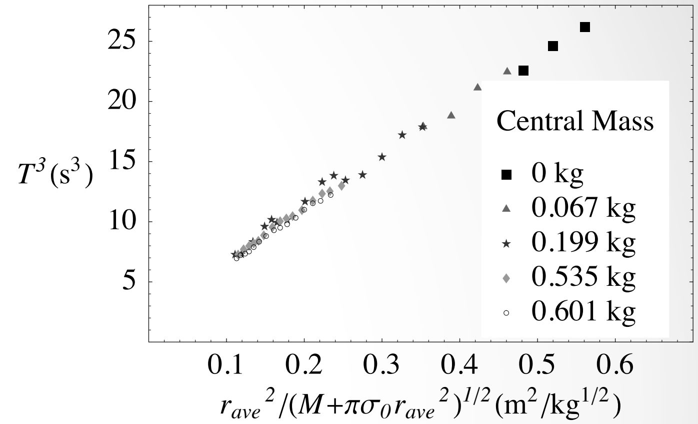
where $2GM/c^2 = 1$

Does a warped spandex fabric yield orbits that obey Kepler's 3 laws?

Circular orbits on a warped spandex fabric, revisited…

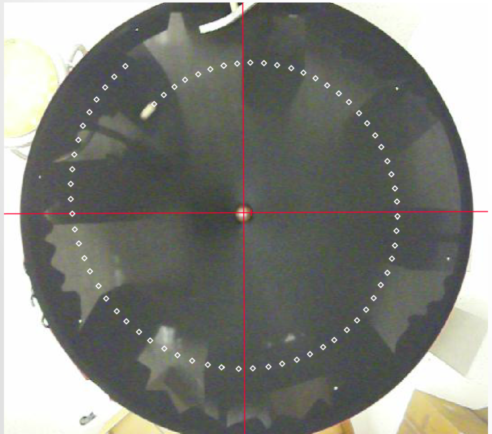
Small slope regime..

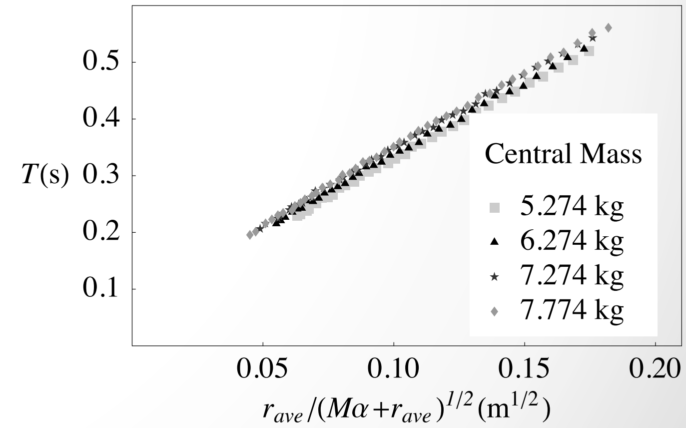
$$T^3 = \left(\frac{28\pi^2}{5g}\right)^{3/2} \frac{1}{\sqrt{2\alpha}} \cdot \frac{r^2}{(M + \pi\sigma_0 r^2)^{1/2}}$$



Large slope regime…

$$T = \left(\frac{28\pi^2}{5g}\right)^{1/2} \cdot \frac{r}{(M\alpha + r)^{1/2}}$$





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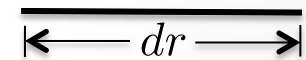
Elastic PE of the *spandex fabric*

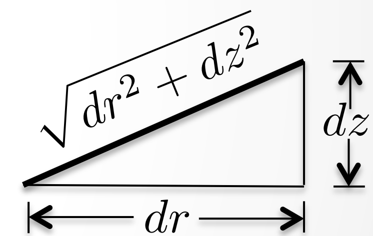
Elastic PE of a *differential concentric ring* of the fabric of unstretched width *dr*…

$$dU_e = \frac{1}{2}\kappa(\sqrt{dr^2 + dz^2} - dr)^2$$



- Define the *modulus of elasticity*, *E* ..

$$E = \frac{\kappa dr}{2\pi r}$$



- Integrating the differential segment over the whole fabric, the *total* elastic PE of the fabric is…

$$U_e = \int_0^R \pi E \cdot r(\sqrt{1 + z'^2} - 1)^2 \, dr$$

Gravitational PE of the *spandex fabric*

Gravitational PE of a *differential concentric ring* of the fabric..

$$dU_{g,s} = dm_s g \cdot z$$

- The mass of the differential ring is a constant under stretching..

$$dm_s = \sigma_0 \cdot 2\pi r dr = \sigma(z') \cdot 2\pi r \sqrt{dr^2 + dz^2}$$

where $\sigma_0, \sigma(z')$ are the *unstretched*, *variable* areal mass densities.

- Integrating the differential segment over the whole fabric, the *total* gravitational PE of the fabric is..

$$U_{g,s} = \int_0^R 2\pi\sigma_0 g \cdot rz \, dr$$

Gravitational PE of the *central mass*

$$U_{g,M} = Mg \cdot z(0) = -\int_0^R Mg \cdot z'(r)\, dr$$

Notice:

- we approximate the central mass as being point-like.

The *total* PE of the spandex-central mass system

$$U = U_e + U_{g,s} + U_{g,M} = \int_0^R f(z, z'; r)\, dr$$

where we defined the functional..

$$f(z, z'; r) \equiv \pi E \cdot r(\sqrt{1 + z'^2} - 1)^2 + 2\pi\sigma_0 g \cdot rz - Mg \cdot z'$$

To *minimize* the *total* PE, subject to the Euler-Lagrange eqn..

$$\frac{\partial f}{\partial z} - \frac{d}{dr}\frac{\partial f}{\partial z'} = 0$$

The shape equation for the elastic fabric

The Euler-Lagrange equation takes the form..

$$\frac{d}{dr}\left[rz'\left[1-\frac{1}{\sqrt{1+z'^2}}\right]-\frac{Mg}{2\pi E}\right]=\frac{\sigma_0 g}{E}\cdot r$$

- which can be integrated..

$$rz'\left[1-\frac{1}{\sqrt{1+z'^2}}\right]=\alpha(M+\pi\sigma_0 r^2)$$

- where we defined the parameter..

$$\alpha \equiv \frac{g}{2\pi E}$$