The orbital motion of marbles on warped 2D surfaces

Chad A. Middleton, Ph.D.

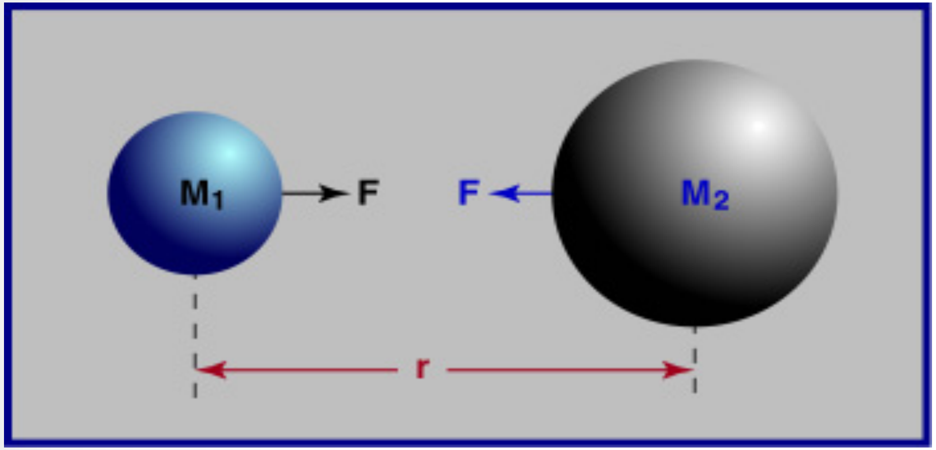
Brown Bag Seminar

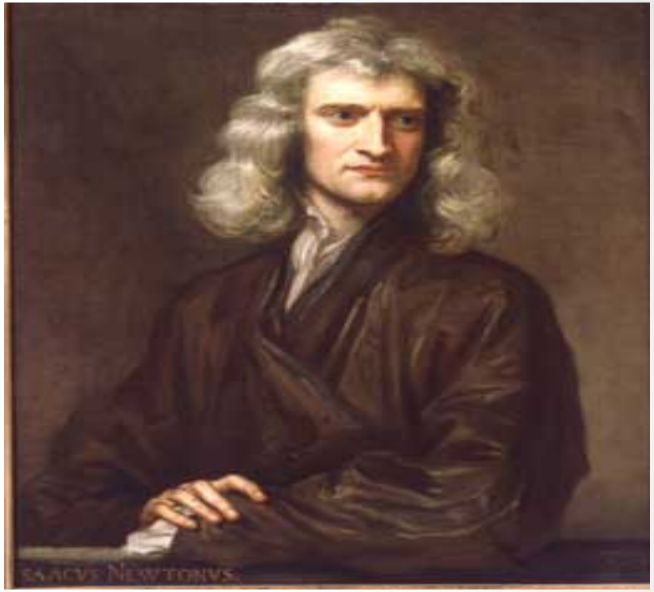
November 7, 2014

Outline

- Gravitation & orbital motion
  - Newtonian gravitation
  - Kepler's 3 laws
  - Einstein's theory of general relativity
- Circular orbits on a warped spandex fabric
- Elliptical-like orbits on a spandex surface
- Newtonian and general relativistic orbits with small eccentricities on 2D surfaces

Newton's universal law of gravitation





$$F = \frac{Gm_1m_2}{r^2}$$

Successes

Described the motion of massive bodies...

...on earth

...in the heavens

Kepler's 3 laws of planetary motion

Kepler's 1st law:

- The planets move in *elliptical orbits* with the Sun at one focus.

Kepler's 2nd law:

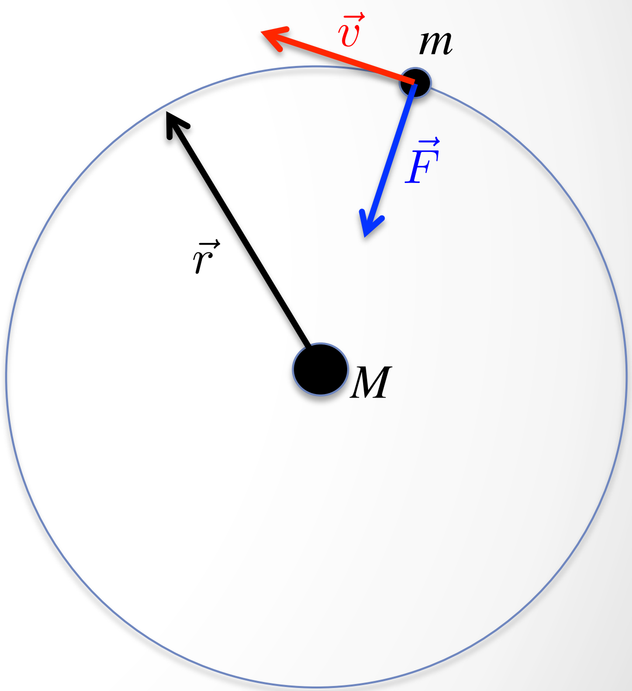
- A line drawn from the Sun to any planet sweeps out *equal areas in equal times*.

Kepler's 3rd law:

- The *period of the planet squared* is proportional to the *radius of its orbit cubed*

or

$$T^2 = \left(\frac{4\pi^2}{G}\right) \cdot \frac{r^3}{M}$$



Notice:

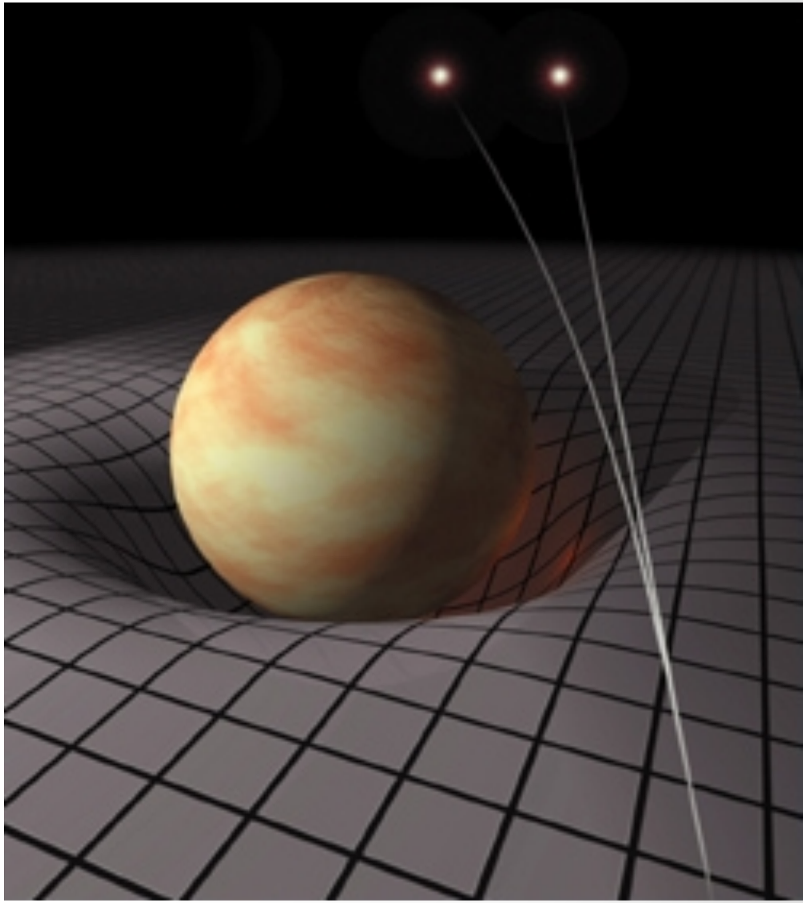
- Kepler's 3rd Law is *independent* of m!

Einstein's theory of general relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $G_{\mu\nu}$ describes the *curvature of spacetime*
- $T_{\mu\nu}$ describes the *matter & energy in spacetime*

Matter tells *space* how to *curve*,
space tells *matter* how to *move*.



Sean M. Carrol, *Spacetime and Geometry: An Introduction to Einstein's General Relativity* (Addison Wesley, 2004)

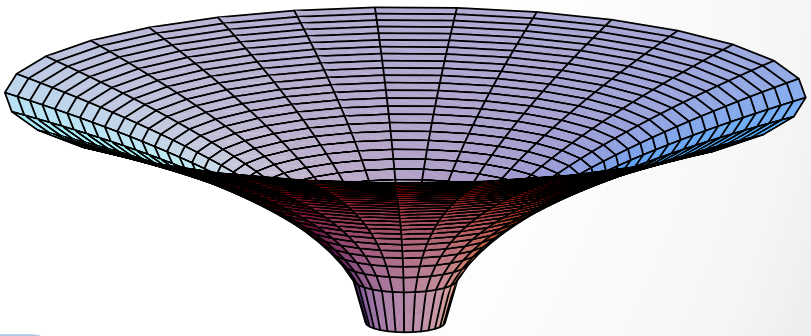
Einstein's theory of general relativity

Consider a *spherically-symmetric, non-rotating massive object…*

Embedding diagram ($t = t_0$, $\theta = \pi/2$)..

- 2D equatorial 'slice' of the 3D space at one moment in time

$$z(r) = 2\sqrt{\frac{2GM}{c^2}\left(r - \frac{2GM}{c^2}\right)}$$



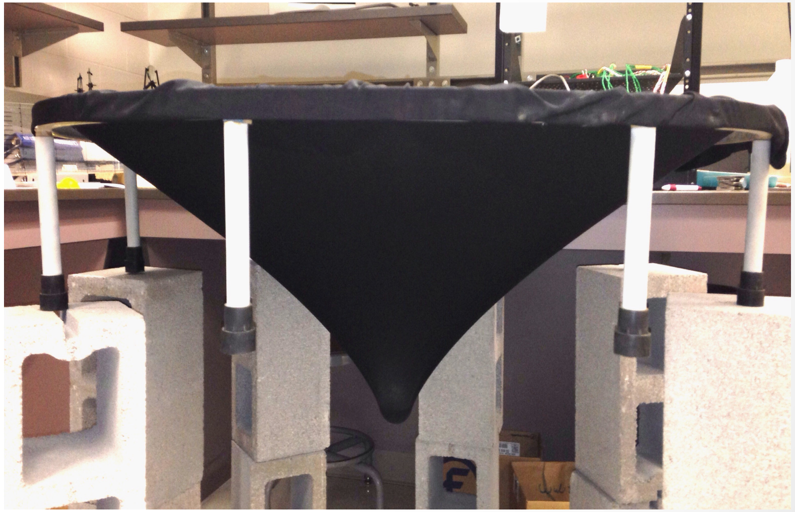
where $2GM/c^2 = 1$

Is there a warped 2D surface that will yield the orbits of planetary motion?

Circular orbits on a warped spandex fabric*

Michael Langston

B.S. Physics, August 2013



*C.A. Middleton, M. Langston, "Circular orbits on a warped spandex fabric," Am. J. Phys. **82** (4), (2014)

The equation of motion for a rolling marble on a *cylindrically-symmetric surface…*

$$(1+z'^2)\frac{d^2r}{d\phi^2} + \left[z'z'' - \frac{2}{r}(1+z'^2)\right]\left(\frac{dr}{d\phi}\right)^2 - r + \frac{7g}{5\ell^2}z'r^4 = 0$$

- in cylindrical coordinates $(r, \phi, z(r))$
- $z' = dz/dr$

The equation of motion for a rolling marble on a *cylindrically-symmetric surface*…

$$(1+z'^2)\frac{d^2r}{d\phi^2}+\left[z'z''-\frac{2}{r}(1+z'^2)\right]\left(\frac{dr}{d\phi}\right)^2-r+\frac{7g}{5\ell^2}z'r^4=0$$

- in cylindrical coordinates $(r, \phi, z(r))$
- $z' = dz/dr$

- For *circular orbits*, the above equation becomes…

$$\frac{4\pi^2 r}{T^2}=\frac{5}{7}g\cdot z'(r)$$

The shape of the spandex fabric

Technique:

1. Construct potential energy (PE) *integral functional* of spandex fabric.
    - *i.* *Elastic* PE of the *spandex*.
    - *ii.* *Gravitational* PE of the *spandex*.
    - *iii.* *Gravitational* PE of the *central mass*.

2. Apply *Calculus of Variations*.

    ⇒ The elastic fabric-mass system will assume the shape which *minimizes* the *total* PE of the system.

i. and *iii*. first considered in "Comment on "The shape of 'the Spandex' and orbits upon its surface", Am. J. Phys. 70 (10), 1056-1058 (2002)

The circular equation of motion & the shape equation

$$\frac{4\pi^2 r}{T^2} = \frac{5}{7} g \cdot z'(r)$$

$$r z' \left[1 - \frac{1}{\sqrt{1 + z'^2}} \right] = \alpha(M + \pi \sigma_0 r^2)$$

- Circular equation of motion
- The shape equation

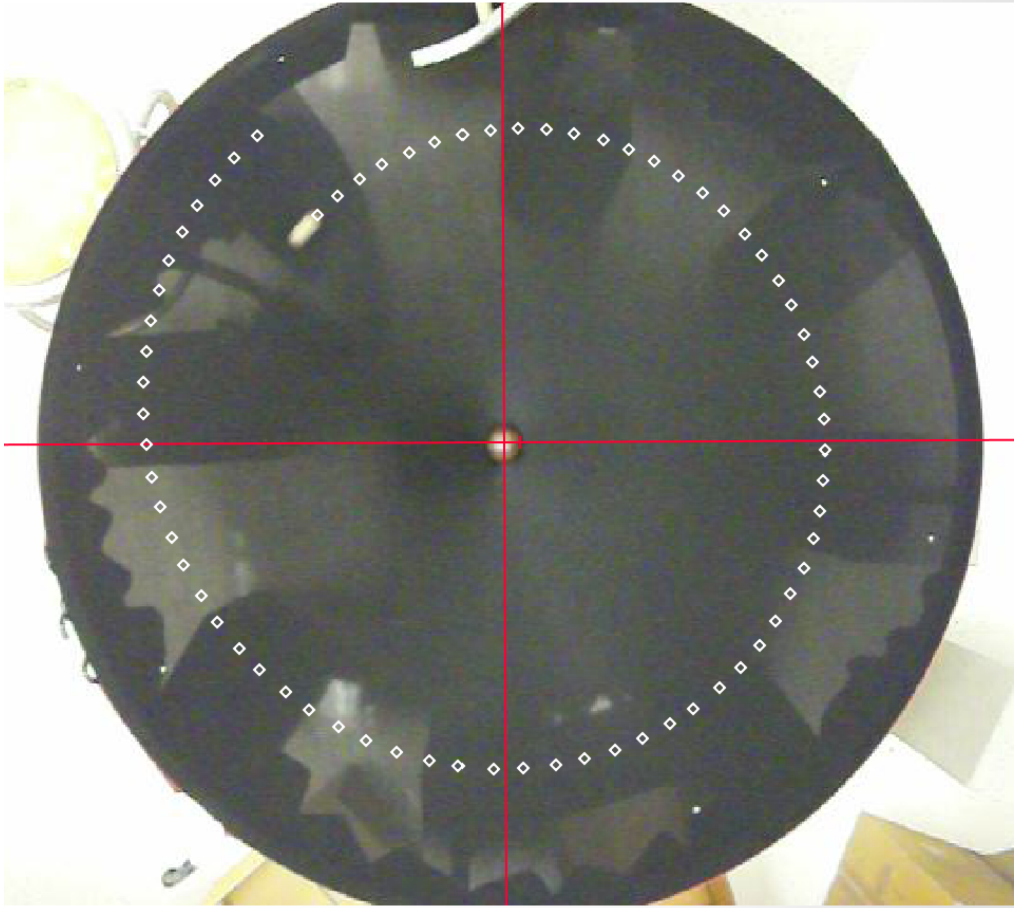
Strategy:

Taylor series expansion in the…

- *small curvature* regime, when $z'(r) \ll 1$
- *large curvature* regime, when $z'(r) \gg 1$

The experimental procedure for circular orbits

- 4 ft. diameter trampoline frame
  - styrofoam insert for *zero* pre-stretch
  - truck tie down around perimeter
- Camera mounted directly above, ramp mounted on frame
- Most *circular* video clip (of ~12) imported into Tracker
- Position determined every 1/30 s and *average radius*, r_{ave}, calculated per revolution
- Shift by 1/8 revolution for subsequent data point
-

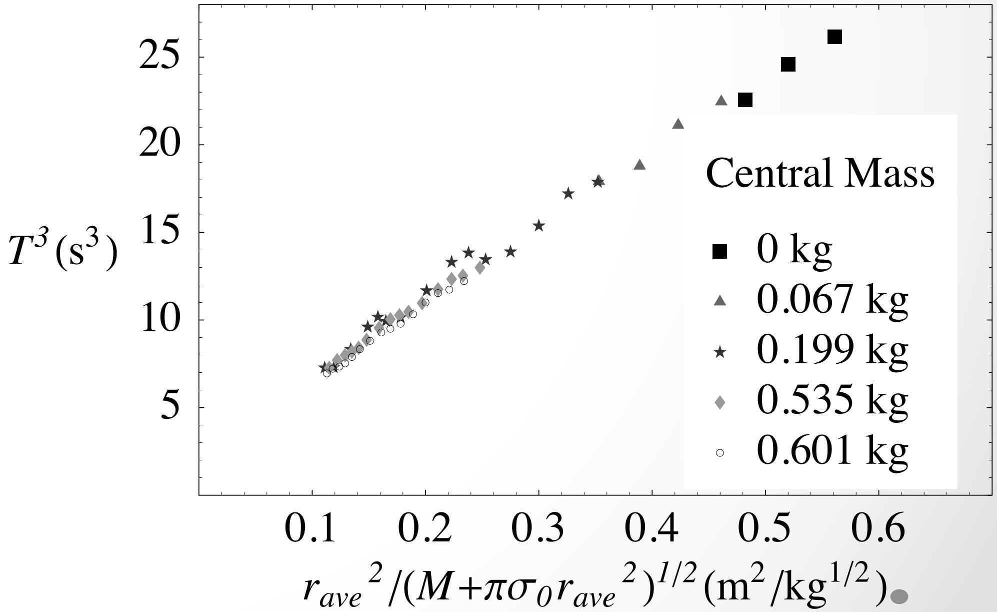


The experiment in the small curvature regime

When $z'(r) \ll 1$ ...

- expanding the shape equation and inserting into the circular eqn of motion

$$T^3 = \left(\frac{28\pi^2}{5g}\right)^{3/2} \frac{1}{\sqrt{2\alpha}} \cdot \frac{r^2}{(M + \pi\sigma_0 r^2)^{1/2}}$$



Notice:

- orbit for *zero* central mass!
- Slope predicts α.

The experiment in the large curvature regime

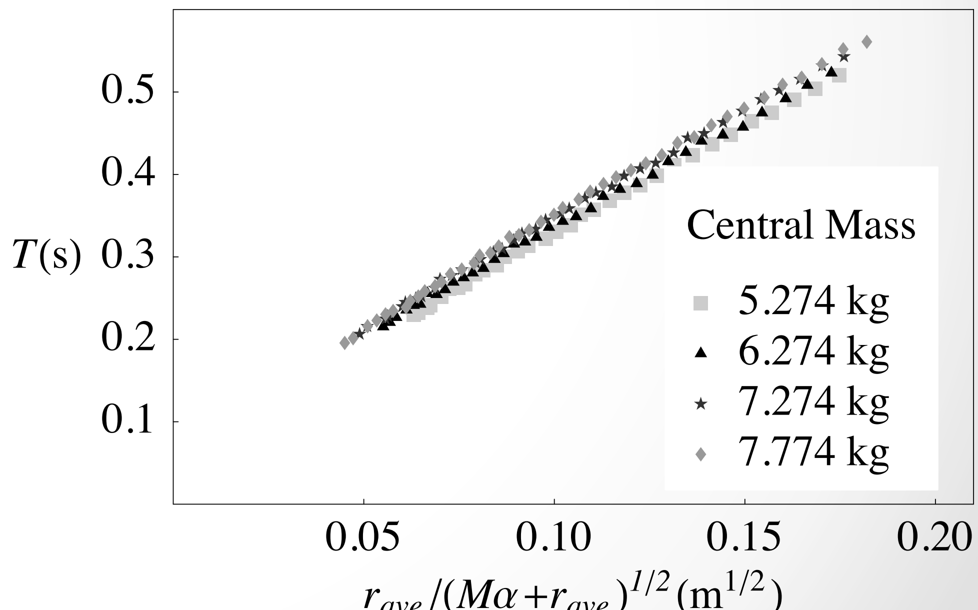
When $z'(r) \gg 1$ ...

- expanding the shape equation and inserting into the circular eqn of motion

$$T = \left(\frac{28\pi^2}{5g}\right)^{1/2} \cdot \frac{r}{(M\alpha + r)^{1/2}}$$

Notice:

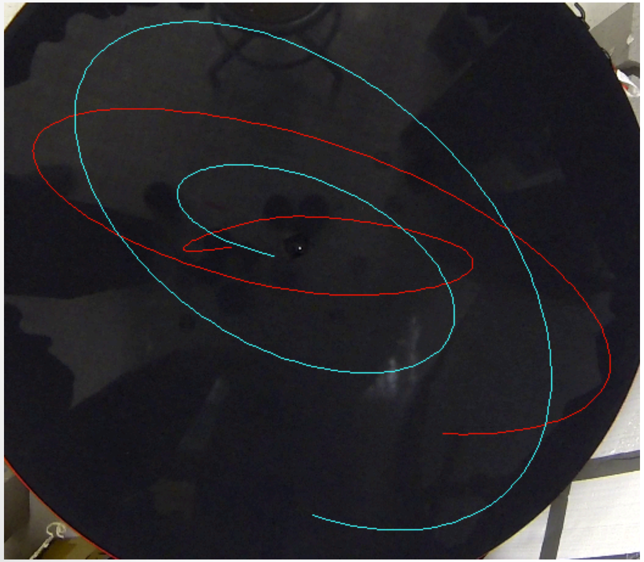
- $\text{slope}_{th} = \left(\frac{28\pi^2}{5g}\right)^{1/2} = 2.37\ \text{s/m}^{1/2}$
- $\text{slope}_{exp} = 2.62\ \text{s/m}^{1/2}$
- ~10% error in the slope.



Elliptical-like orbits on a Spandex surface*

Danny Weller

B.S. Physics, December 2014



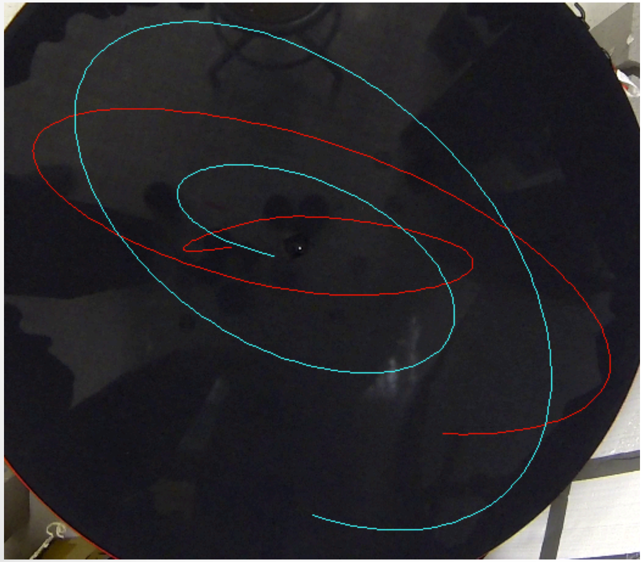
- *CMU Physics Seminar on Thursday, November 20, 12:30-1:30pm in WS 202

Elliptical-like orbits on a Spandex surface*

Danny Weller

B.S. Physics, December 2014

(assuming he passes Modern Optics!)



*CMU Physics Seminar on Thursday, November 20, 12:30-1:30pm in WS 202

The equation of motion for a rolling marble on a *cylindrically-symmetric surface…*

$$(1+z'^2)\frac{d^2r}{d\phi^2} + \left[z'z'' - \frac{2}{r}(1+z'^2)\right]\left(\frac{dr}{d\phi}\right)^2 - r + \frac{7g}{5\ell^2}z'r^4 = 0$$

For elliptical orbits with small eccentricities…

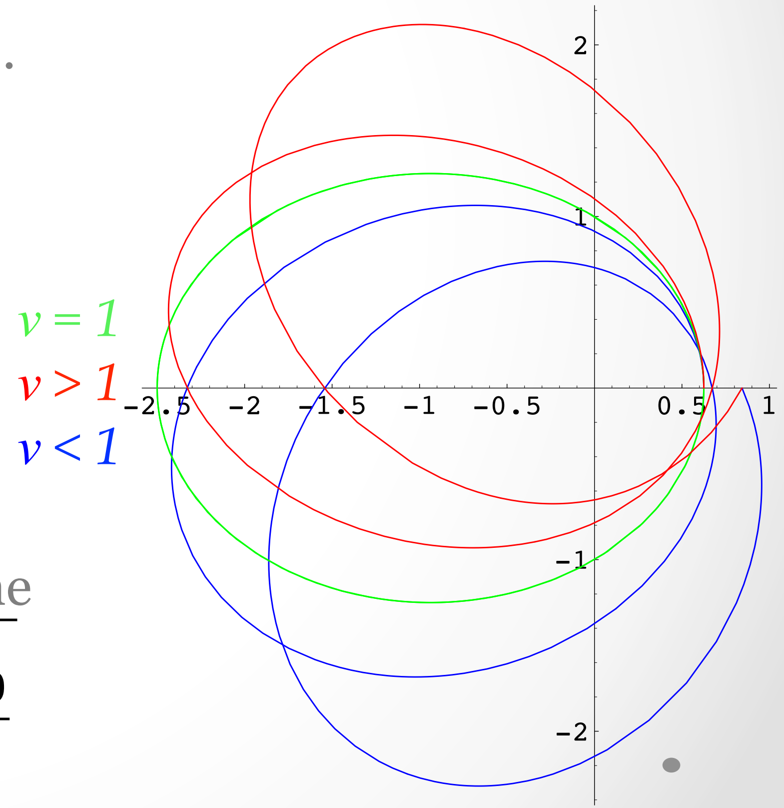
$$r(\phi) = r_0(1 - \varepsilon\cos(\nu\phi))$$

- where ν is the *precession parameter*

$$\nu \equiv \frac{2\pi}{\Delta\phi}$$

- Inserting the approximate solution into the equation of motion yields

$$\nu = \sqrt{\frac{3z_0' + z_0''r_0}{z_0' + z_0'^3}}$$



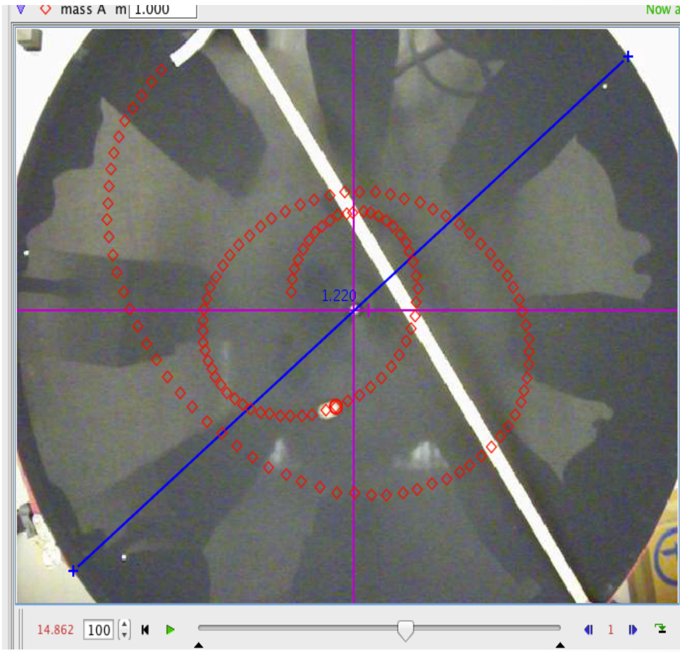
The experimental procedure for elliptical orbits

- Position determined every 1/60 s and *average radius*, r_{ave}, calculated per $\Delta\phi$.

- *Angular displacement* from r_{max} to r_{max} is measured and the *precession parameter* is calculated from

$$\nu = \frac{2\pi}{\Delta\phi} \quad ,$$

- which can be compared to the theoretical value.

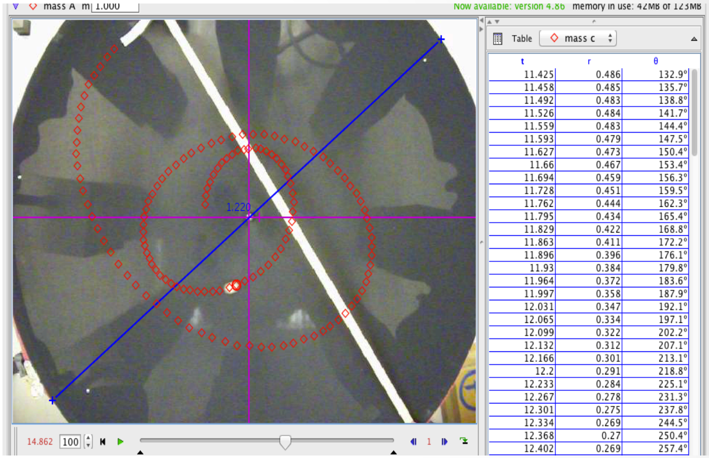


t	r	θ
11.425	0.486	132.9°
11.458	0.485	135.7°
11.492	0.483	138.8°
11.526	0.484	141.7°
11.559	0.483	144.4°
11.593	0.479	147.5°
11.627	0.473	150.4°
11.66	0.467	153.4°
11.694	0.459	156.3°
11.728	0.451	159.5°
11.762	0.444	162.3°
11.795	0.434	165.4°
11.829	0.422	168.8°
11.863	0.411	172.2°
11.896	0.396	176.1°
11.93	0.384	179.8°
11.964	0.372	183.6°
11.997	0.358	187.9°
12.031	0.347	192.1°
12.065	0.334	197.1°
12.099	0.322	202.2°
12.132	0.312	207.1°
12.166	0.301	213.1°
12.2	0.291	218.8°
12.233	0.284	225.1°
12.267	0.278	231.3°
12.301	0.275	237.8°
12.334	0.269	244.5°
12.368	0.27	250.4°
12.402	0.269	257.4°

The experiment in the small curvature regime

- For $z'(r) \ll 1$, the slope of the spandex surface takes the form…

$$z'(r) \simeq \left(\frac{2\alpha}{r}\right)^{1/3} (M + \sigma_0 \pi r^2)^{1/3} .$$



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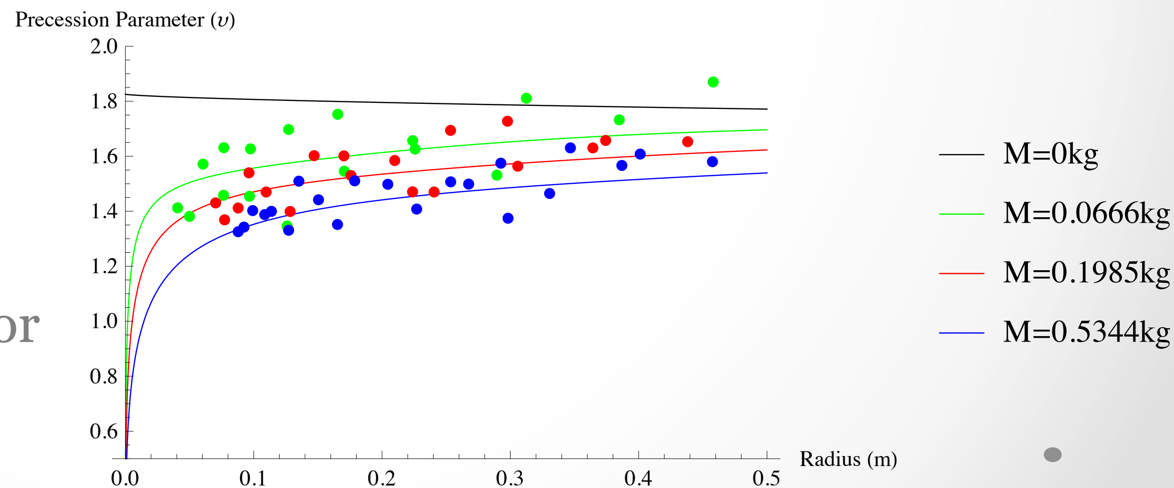
- Plugging this slope into the equation determining the *precession parameter*

$$\nu = \sqrt{\frac{3z_0' + z_0'' r_0}{z_0' + z_0'^3}}$$

- yields a theoretical value for

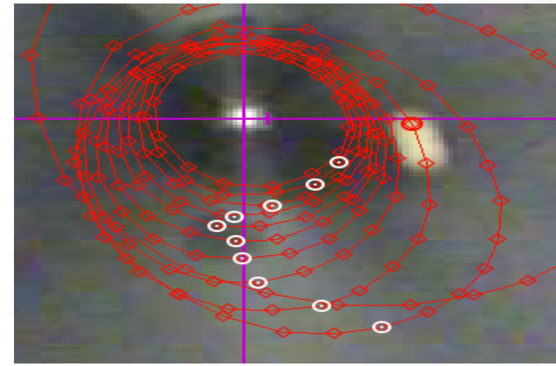
$$\nu = \nu(M, r)$$



The experiment in the large curvature regime

- For $z'(r) \gg 1$, the slope of the spandex surface takes the form…

$$z'(r) \simeq \frac{M\alpha}{r} + 1$$



- Plugging this slope into the equation determining the *precession parameter*

$$\nu = \sqrt{\frac{3z_0' + z_0'' r_0}{z_0' + z_0'^3}}$$

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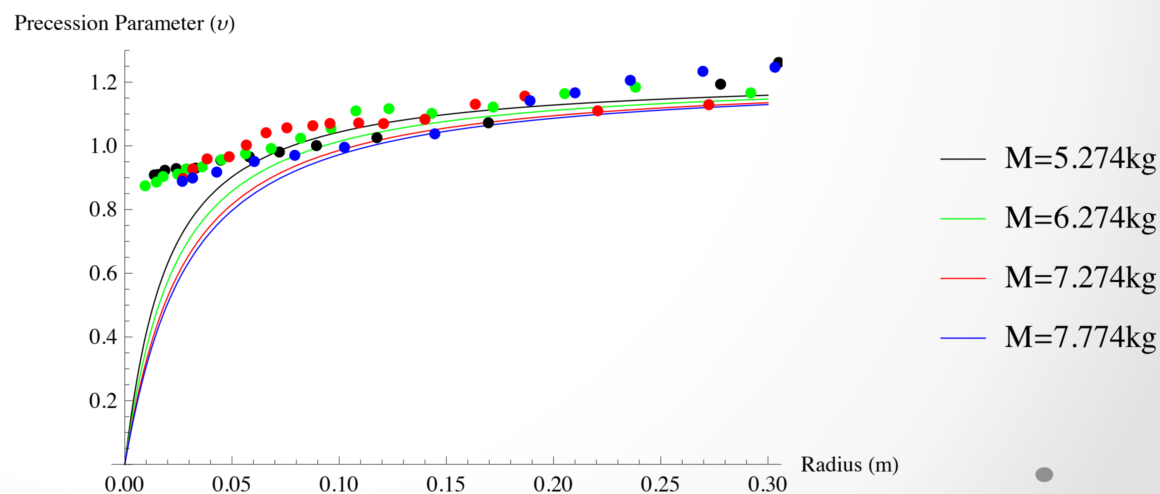
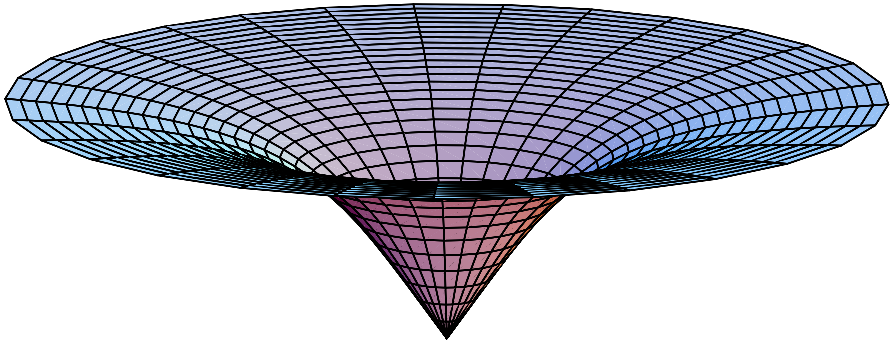


Notice:

- $\nu < 1$ for small r!

Newtonian and general relativistic orbits with small eccentricities on 2D surfaces



submitted to the *American Journal of Physics*

Is there a 2D surface that will yield planetary orbits?

Compare the equation of motion for the 2D *orbiting object*

$$(1+z'^2)\frac{d^2r}{d\phi^2}+\left[z'z''-\frac{2}{r}(1+z'^2)\right]\left(\frac{dr}{d\phi}\right)^2-r+\frac{\tilde{g}}{\tilde{\ell}^2}z'r^4=0 \quad *$$

- to that of an *object in a Newtonian potential*

$$\frac{d^2r}{d\phi^2}-\frac{2}{r}\left(\frac{dr}{d\phi}\right)^2-r+\frac{GM}{\ell^2}r^2=0$$

* will NOT yield *Newtonian orbits* on ANY cylindrically-symmetric surface, *except* in the special case of circular orbits.

L.Q. English and A. Mareno, "Trajectories of rolling marbles on various funnels", Am. J. Phys. **80** (11), 996-1000 (2012)

The equation of motion for an object constrained to reside on a *cylindrically-symmetric surface*…

$$(1+z'^2)\frac{d^2r}{d\phi^2}+\left[z'z''-\frac{2}{r}(1+z'^2)\right]\left(\frac{dr}{d\phi}\right)^2-r+\frac{\tilde{g}}{\tilde{\ell}^2}z'r^4=0$$

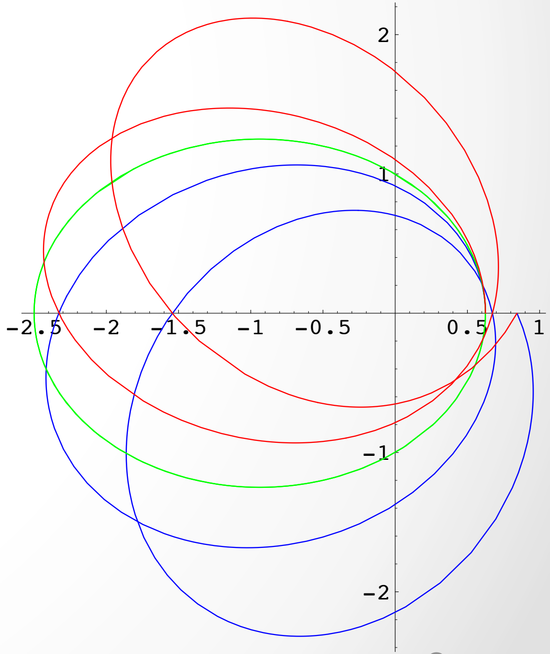
- For orbits with small eccentricities…

$$r(\phi)=r_0(1-\varepsilon\cos(\nu\phi))$$

- We find a valid solution, to 1st order in the ε, when

$$\tilde{\ell}^2=\tilde{g}r_0^3z_0'$$

$$z_0'(1+z_0'^2)\nu^2=3z_0'+r_0z_0''$$

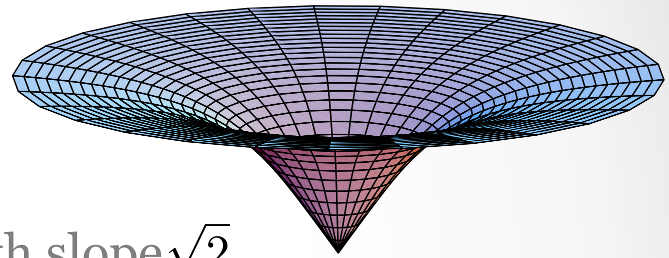


Michael Nauenberg, "Perturbation approximation for orbits in axially symmetric funnels", *to appear in* Am. J. Phys.

2D Surfaces…

2D surface that generates Newtonian *stationary* elliptical orbits..

$$\frac{dz}{dr} = \sqrt{2}\,(1+\kappa r^4)^{-1/2} \quad *$$



- When $\kappa = 0$, * yields eqn for an *inverted cone* with slope $\sqrt{2}$.

2D surface that generates GR *precessing* elliptical orbits…

$$\frac{dz}{dr} = \sqrt{\frac{2+\beta}{1-\beta}} \cdot (1+\kappa r^{2(2+\beta)})^{-1/2} \quad **$$

where $\beta = 6GM/c^2 r_0$

Planets	r_o (m)	ε	β
Mercury	$5.79*10^{10}$	0.2056	$1.53*10^{-7}$
Venus	$1.08*10^{10}$	0.0068	$8.19*10^{-7}$
Earth	$1.50*10^{11}$	0.0167	$5.90*10^{-8}$
Mars	$2.28*10^{11}$	0.0934	$3.88*10^{-8}$
Jupiter	$7.78*10^{11}$	0.0483	$1.14*10^{-8}$
Saturn	$1.43*10^{12}$	0.056	$6.19*10^{-9}$
Uranus	$2.87*10^{12}$	0.0461	$3.08*10^{-9}$
Neptune	$4.50*10^{12}$	0.01	$1.97*10^{-9}$

Notice:

- They agree when $\beta \to 0$; ** diverges when $\beta \to 1$.
- GR offers a *tiny* correction for the orbits of the solar system planets.

Gary D. White, "On trajectories of rolling marbles in cones and other funnels", Am. J. Phys. 81 (12), 890-898 (2013)

Direct measurement of the *modulus of elasticity, E*

The *shape equation* in the large curvature regime is..

$$z'(r) \simeq \frac{M\alpha}{r} + 1$$

- integrating yields..

$$\frac{z(M)}{\ln(R_B)} = (M - M_0)\alpha$$

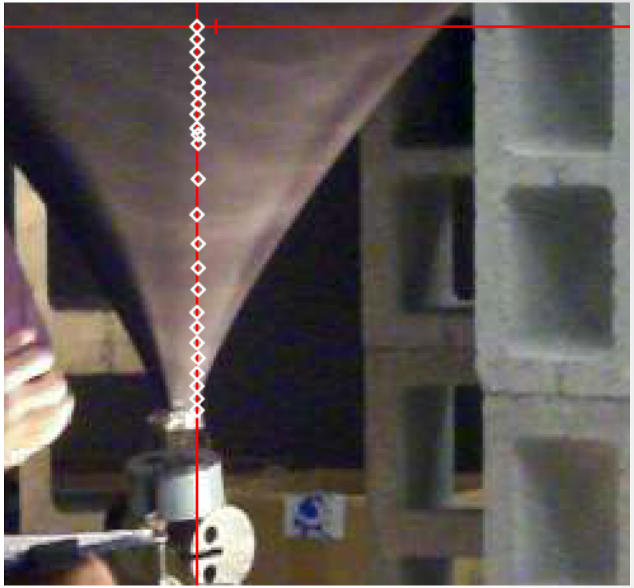
Notice:

- $M = 0.274 - 0.674$ kg regime

$$\alpha \simeq 0.030 \text{ m/kg}$$

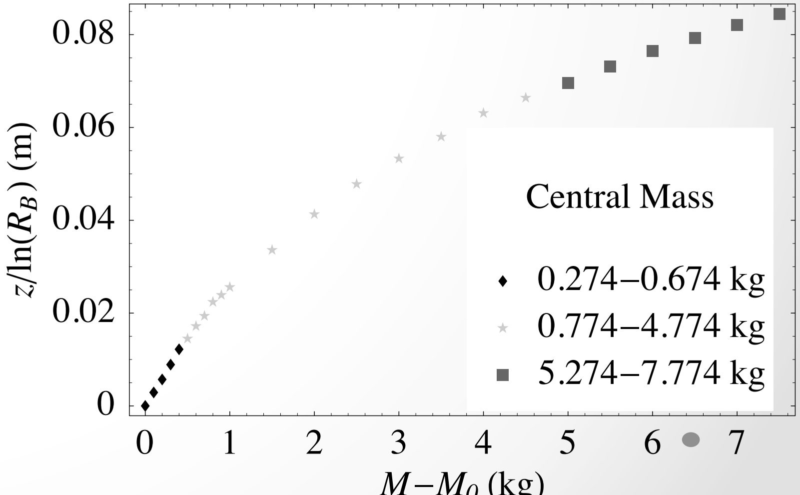
- $M = 5.274 - 7.774$ kg regime

$$\alpha \simeq 0.006 \text{ m/kg}$$



Top 10: $M = 0.274$kg - 1.174kg in 0.1 kg intervals

Bottom 14: $M = 1.274$kg - 7.774kg in 0.5 kg intervals



Circular orbits in general relativity

One arrives at an *exact* Kepler-like expression of the form..

$$T^2 \propto \frac{r^3}{(M - 2 \cdot \rho_0 \cdot 4\pi r^3/3)}$$ *

- Kepler's 3rd Law when $\rho_0 = 0$.

Compare to the Kepler-like relation for a marble on the warped spandex fabric in the small curvature regime..

$$T^3 \propto \frac{r^2}{(M + \sigma_0 \cdot \pi r^2)^{1/2}}$$

- Areal mass density, σ_0, plays the role of a *negative* vacuum energy, ρ_o.

*N. Cruz, M. Olivares, and J. Villanueva, "The geodesic structure of the Schwarzschild Anti-de Sitter black hole", Classical and Quantum Gravity **22**, 1167 (2005)